Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 5 Due March 14, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 27

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1. (4 marks) Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{c(10-x)^4}{\sqrt{x-1}} & \text{for } 1 < x < 10 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a normalizing constant. Find c and $E[X]$. *Hint:* Write $X = 1 + 9Y$, where Y has a Beta distribution.

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2. (8 marks)

(a) (3 marks) For $\alpha, \beta > 0$, show that

$$B(\alpha, \beta) = 2 \int_0^{\infty} t^{2\alpha-1}(1+t^2)^{-(\alpha+\beta)} dt.$$

Hint: Make the substitution $x = t^2/(1+t^2)$ in

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx.$$

(b) (5 marks) Refer back to Problem 2 on Homework 4. For the t distribution with n degrees of freedom, the mean does not exist for $n = 1$ and the variance does not exist for $n \leq 2$. For $n \geq 2$ the mean is 0. For $n \geq 3$, use part(a) to find the variance of the t distribution with n degrees of freedom.

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3. (5 marks) A fair die is rolled ten times.

(a) (2 marks) What is the probability that the number of 1's plus the number of 2's equals three and the number of 3's equals four?

(b) (3 marks) Given that exactly four of the ten rolls resulted in an outcome less than 4, what is the probability that three 5's were rolled. (Recall that for 2 events A and B with $P(B) > 0$, the conditional probability of A given B is $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$).

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4. (5 marks) Let $X_1, \ldots, X_r$ be independent Poisson random variables, with X_i distributed as Poisson(λ_i), $i = 1, \ldots, r$. Let n be a fixed positive integer. Compute

$$P(X_1 = x_1, \ldots, X_r = x_r \mid \sum_{i=1}^{r} X_i = n)$$

for all $(x_1, \ldots, x_r)^T \in \mathbb{R}^r$. You may use the fact that $\sum_{i=1}^{r} X_i$ has a Poisson($\lambda_1 + \ldots + \lambda_r$) distribution.

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5. (5 marks) Let $(X_1, \ldots, X_k)$ have a multinomial distribution with parameters n and $p_1, \ldots, p_k$. For $i, j = 1, \ldots, k$, find $E[X_i X_j]$. *Hint:* Write $X_i = X_{i1} + \ldots + X_{in}$ where

$$X_{ik} = \begin{cases} 1 & \text{if the } k\text{th multinomial experiment has outcome } i \\ 0 & \text{otherwise,} \end{cases}$$

and similarly write $X_j = X_{j1} + \ldots + X_{jn}$.