Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Homework 5 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 27

Student Number Name

1. (4 marks) Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{c(10-x)^4}{\sqrt{x-1}} & \text{for } 1 < x < 10 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a normalizing constant. Find c and $E[X]$.

Solution: From the lecture notes on the Beta distribution we can deduce that X is equal in distribution to $1 + 9Y$, where Y has a Beta distribution with parameters $\frac{1}{2}$ and 5. The normalizing constant of this Beta distribution is

$$\frac{\Gamma(1/2 + 5)}{\Gamma(1/2)\Gamma(5)} = \frac{52.34278}{(\sqrt{\pi})(4!)} = 1.230469.$$

where $\Gamma(5.5) = (4.5)(3.5)(2.5)(1.5)(.5)\Gamma(.5) = 29.53125\sqrt{\pi} = 52.34278$, using the recursive property of the Gamma function (or see Problem 3 of Homework 4). The normalizing constant of the distribution of X is then

$$c = \frac{1.230469}{9^{4.5}} = \frac{1.230469}{19683} = .0000625143.$$

(There was a typo in the lecture notes for the 2nd lecture of Week 7 where I forgot to put the factor $\frac{1}{b-a}$ in the pdf of Y. It is fixed now. If you use the pdf for Y that had the typo then you get the normalizing constant to be

$$c = \frac{1.230469}{9^{3.5}} = \frac{1.230469}{2187} = .00056263.$$

Either answer for c is acceptable for full marks). Using the relationship $X = 1 + 9Y$ we have

$$E[X] = 1 + 9E[Y] = 1 + 9\frac{1/2}{1/2 + 5} = 1 + \frac{9}{11} = \frac{20}{11} = 1.818.$$

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2. (8 marks)

(a) (3 marks) For $\alpha, \beta > 0$, show that

$$B(\alpha,\beta) = 2\int_0^\infty t^{2\alpha-1}(1+t^2)^{-(\alpha+\beta)}dt.$$

Hint: Make the substitution $x = t^2/(1+t^2)$ in

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

(b) (5 marks) Refer back to Problem 2 on Homework 4. For the t distribution with n degrees of freedom, the mean does not exist for $n = 1$ and the variance does not exist for $n \leq 2$. For $n \geq 2$ the mean is 0. For $n \geq 3$, use part(a) to find the variance of the t distribution with n degrees of freedom.

Solution:

(a) We have $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$. Making the substitution $x = t^2/(1+t^2)$,

$$dx = \frac{(1+t^2)2t - t^2(2t)}{(1+t^2)^2}dt = \frac{2t}{(1+t^2)^2}dt,$$

and $t = \sqrt{x/(1-x)}$, so that as x varies between 0 and 1, the limits of t are from 0 to ∞. Thus, we obtain

$$B(\alpha,\beta) = \int_0^\infty \left(\frac{t^2}{1+t^2}\right)^{\alpha-1}\left(\frac{1}{1+t^2}\right)^{\beta-1}\frac{2t}{(1+t^2)^2}dt = 2\int_0^\infty t^{2\alpha-1}(1+t^2)^{-(\alpha+\beta)}dt.$$

(b) Let $n \geq 3$ and let $U \sim t_n$. Then

$$\begin{aligned}
\mathrm{Var}(U) &= E[U^2] = \int_{-\infty}^\infty u^2 \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{u^2}{n}\right)^{-(n+1)/2} du \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} 2\int_0^\infty u^2\left(1+\frac{u^2}{n}\right)^{-(n+1)/2} du
\end{aligned}$$

Making the substitution $t = u/\sqrt{n}$, we have $u = \sqrt{n}t$, $du = \sqrt{n}dt$, and the limits for t remain from 0 to ∞. Then we have

$$\begin{aligned}
\mathrm{Var}(U) &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} 2\int_0^\infty nt^2\left(1+\frac{nt^2}{n}\right)^{-(n+1)/2}\sqrt{n}dt \\
&= \frac{n\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} 2\int_0^\infty t^2(1+t^2)^{-(n+1)/2}dt
\end{aligned}$$

Setting $2\alpha-1=2$ and $\alpha+\beta=(n+1)/2$ in part(a), we get $\alpha=3/2$ and $\beta=(n-2)/2$, which is positive for $n \geq 3$. With these values of α and β and using part(a), we have

$$\begin{aligned}
\mathrm{Var}(U) &= \frac{n\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} B\left(\frac{3}{2},\frac{n-2}{2}\right) \\
&= \frac{n\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{n-2}{2}\right)} \\
&= \frac{n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\frac{n-2}{2}\Gamma\left(\frac{n-2}{2}\right)} \\
&= \frac{n\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}\frac{n-2}{2}} = \frac{n}{n-2},
\end{aligned}$$

since $\Gamma(1/2)=\sqrt{\pi}$.

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3. (5 marks) A fair die is rolled ten times.

(a) (2 marks) What is the probability that the number of 1's plus the number of 2's equals three and the number of 3's equals four?

(b) (3 marks) Given that exactly four of the ten rolls resulted in an outcome less than 4, what is the probability that three 5's were rolled. (Recall that for 2 events A and B with $P(B) > 0$, the conditional probability of A given B is $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$).

Solution:

(a) Let X_{12} be the number of 1's or 2's, X_3 the number of 3's, and X_{456} the number of 4's, 5's or 6's. Then the vector (X_{12}, X_3, X_{456}) has a Multinomial distribution with parameters 10 and $p_{12} = 1/3$, $p_3 = 1/6$, $p_{456} = 1/2$. The desired probability is

$$\begin{aligned}
P(X_{12} = 3, X_3 = 4) &= P(X_{12} = 3, X_3 = 4, X_{456} = 3) \\
&= \frac{10!}{3!4!3!}\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^4\left(\frac{1}{2}\right)^3 \approx 0.015.
\end{aligned}$$

(b) Let X_{123} be the number of 1's, 2's, or 3's, X_{46} the number of 4's or 6's, and X_5 the number of 5's. Then (X_{123}, X_{46}, X_5) has a Multinomial distribution with parameters 10 and $p_{123} = 1/2$, $p_{46} = 1/3$, $p_5 = 1/6$. The desired conditional probability is

$$\begin{aligned}
P(X_5 = 3 \mid X_{123} = 4) &= \frac{P(X_5 = 3, X_{123} = 4)}{P(X_{123} = 4)} \\
&= \frac{P(X_5 = 3, X_{123} = 4, X_{46} = 3)}{P(X_{123} = 4)} \\
&= \frac{\frac{10!}{3!4!3!}(1/6)^3(1/2)^4(1/3)^3}{\frac{10!}{4!6!}(1/2)^4(1/2)^6} \\
&= \frac{6!}{3!3!}\frac{2^3}{3^6} = 20 \times \frac{8}{729} = \frac{160}{729} = .2195.
\end{aligned}$$

(We are using the fact that $P(X_5 = 3, X_{123} = 4) = P(X_5 = 3, X_{123} = 4, X_{46} = 3)$ since $\{X_5 = 3, X_{123} = 4\}$ implies $\{X_{46} = 3\}$, and the marginal distribution of X_{123} is Binomial$(10, \frac{1}{2})$).

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4. (5 marks) Let $X_1, \ldots, X_r$ be independent Poisson random variables, with X_i distributed as Poisson(λ_i), $i = 1, \ldots, r$. Let n be a fixed positive integer. Compute

$$P(X_1 = x_1, \ldots, X_r = x_r \mid \sum_{i=1}^{r} X_i = n)$$

for all $(x_1, \ldots, x_r)^T \in \mathbb{R}^r$. You may use the fact that $\sum_{i=1}^r X_i$ has a Poisson$(\lambda_1 + \ldots + \lambda_r)$ distribution.

Solution: We have

$$P\Big(X_1 = x_1, \ldots, X_r = x_r \;\Big|\; \sum_{i=1}^{r} X_i = n\Big) = \frac{P\big(X_1 = x_1, \ldots, X_r = x_r, \sum_{i=1}^r X_i = n\big)}{P\big(\sum_{i=1}^r X_i = n\big)}.$$

First, we note that the above conditional probability is nonzero if and only if each of $x_1, \ldots, x_r$ is a nonnegative integer and $\sum_{i=1}^r x_i = n$ (since otherwise the numerator on the right hand side above is 0). For $x_1, \ldots, x_r$ nonnegative integers satisfying $\sum_{i=1}^r x_i = n$, we have

$$\begin{aligned}
\frac{P\big(X_1 = x_1, \ldots, X_r = x_r, \sum_{i=1}^r X_i = n\big)}{P\big(\sum_{i=1}^r X_i = n\big)} &= \frac{P(X_1 = x_1, \ldots, X_r = x_r)}{P\big(\sum_{i=1}^r X_i = n\big)} \\
&= \frac{P(X_1 = x_1) \ldots P(X_r = x_r)}{P\big(\sum_{i=1}^r X_i = n\big)} \\
&= \frac{(\lambda_1^{x_1}/x_1!)e^{-\lambda_1} \ldots (\lambda_r^{x_r}/x_r!)e^{-\lambda_r}}{((\lambda_1 + \ldots + \lambda_r)^n/n!)e^{-(\lambda_1 + \ldots + \lambda_r)}} \\
&= \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \ldots p_r^{x_r},
\end{aligned}$$

where $p_i = \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_r}$ for $i = 1, \ldots, r$. Thus we see that the conditional distribution of $(X_1, \ldots X_r)^T$ given $\sum_{i=1}^r X_i = n$ is Multinomial with parameters n and $p_1, \ldots, p_r$.

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5. (5 marks) Let $(X_1, \ldots, X_k)$ have a multinomial distribution with parameters n and $p_1, \ldots, p_k$. For $i, j = 1, \ldots, k$, find $E[X_i X_j]$. *Hint:* Write $X_i = X_{i1} + \ldots + X_{in}$ where

$$X_{ik} = \begin{cases} 1 & \text{if the } k\text{th multinomial experiment has outcome } i \\ 0 & \text{otherwise,} \end{cases}$$

and similarly write $X_j = X_{j1} + \ldots + X_{jn}$.

Solution: First suppose that $i \neq j$. Expressing X_i and X_j as in the hint, we have

$$\begin{aligned} E[X_i X_j] &= E[(X_{i1} + \ldots + X_{in})(X_{j1} + \ldots + X_{jn})] \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} E[X_{ik} X_{j\ell}] \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} P(\text{trial } k \text{ has outcome } i \text{ and trial } \ell \text{ has outcome } j) \\ &= \sum_{k=1}^{n} \sum_{\ell \neq k} P(\text{trial } k \text{ has outcome } i \text{ and trial } \ell \text{ has outcome } j), \end{aligned}$$

where the last equality follows because a given trial cannot have both outcome i and outcome j. Since the trials are independent, we have

$$\begin{aligned} E[X_i X_j] &= \sum_{k=1}^{n} \sum_{\ell \neq k} P(\text{trial } k \text{ has outcome } i) P(\text{trial } \ell \text{ has outcome } j) \\ &= \sum_{k=1}^{n} \sum_{\ell \neq k} p_i p_j = n(n-1) p_i p_j. \end{aligned}$$

If $i = j$ then $E[X_i X_j] = E[X_i^2] = \text{Var}(X_i) + E[X_i]^2$. But X_i has a Binomial(n, p_i) distribution, so $E[X_i^2] = np_i(1 - p_i) + (np_i)^2 = np_i(1 - p_i + np_i) = np_i(1 + (n-1)p_i)$.