Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 6 Due March 21, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 30

Student Number ____________ Name ____________

1. (9 marks) Urn 1 contains n red balls and urn 2 contains n blue balls. At each stage a ball is randomly chosen from urn 1 and a second ball is randomly chosen from urn 2, then the ball from urn 1 is placed into urn 2 and the ball from urn 2 is placed into urn 1. This is a simple diffusion model described by Daniel Bernoulli (1769). Let X denote the number of red balls in urn 1 after k stages.

(a) (4 marks) Show, using the binomial theorem, that

$$\sum_{\substack{m=0: \\ m \text{ even}}}^{k} \binom{k}{m} \left(\frac{1}{n}\right)^m \left(1-\frac{1}{n}\right)^{k-m} = \frac{1}{2}\left\{\left[\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]^k + \left[\left(1-\frac{1}{n}\right)-\frac{1}{n}\right]^k\right\}.$$

(b) (5 marks) Find $E[X]$. Hint: label the red balls $1, \ldots, n$ and define

$$X_i = \begin{cases} 1 & \text{if the red ball labelled } i \text{ is in urn 1 after the } k\text{th stage} \\ 0 & \text{otherwise.} \end{cases}$$

The red ball labelled 1 will be in urn 1 after k stages if and only if it was drawn an even number of times.

Student Number Name

2. (5 marks) Let $X_1, X_2, \ldots$ be a sequence of continuous, independent, and identically distributed random variables. Let

$$N = \min\{n : X_1 \geq X_2 \geq X_3 \geq \ldots \geq X_{n-1}, X_{n-1} < X_n\}.$$

Find $E[N]$. *Hint:* Compute $E[N]$ as $\sum_{n=1}^{\infty} P(N \geq n)$.

Student Number Name

3. (5 marks) Let X be a random variable and let $f(\cdot)$ and $g(\cdot)$ be nondecreasing bounded functions. In this problem we wish to show that $\mathrm{Cov}(f(X), g(X)) \geq 0$. To do this let X' be a random variable independent of X and with the same distribution as X. First, show that

$$\mathrm{Cov}(f(X), g(X)) = \frac{1}{2}\mathrm{Cov}(f(X) - f(X'), g(X) - g(X')).$$

Deduce from the above equality that $\mathrm{Cov}(f(X), g(X)) \geq 0$.

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4. (5 marks) Let $(X_1, \ldots, X_k)^T$ have a Multinomial distribution with parameters n and $p_1, \ldots, p_k$. Find the correlation between X_i and X_j for any $i, j = 1, \ldots, k$. (cf. Problem 5 on Homework 5).

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5. (6 marks) Let $X_1, \ldots, X_n$ be independent $U(0,1)$ random variables and let $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$ denote the first and last order statistics. Find $\rho(X_{(1)}, X_{(n)})$, the correlation coefficient between $X_{(1)}$ and $X_{(n)}$.