Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Homework 6 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 30

Student Number Name

1. (9 marks) Urn 1 contains n red balls and urn 2 contains n blue balls. At each stage a ball is randomly chosen from urn 1 and a second ball is randomly chosen from urn 2, then the ball from urn 1 is placed into urn 2 and the ball from urn 2 is placed into urn 1. This is a simple diffusion model described by Daniel Bernoulli (1769). Let X denote the number of red balls in urn 1 after k stages.

(a) (4 marks) Show, using the binomial theorem, that

$$\sum_{\substack{m=0: \\ m \text{ even}}}^{k} \binom{k}{m} \left(\frac{1}{n}\right)^m \left(1-\frac{1}{n}\right)^{k-m} = \frac{1}{2}\left\{\left[\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]^k + \left[\left(1-\frac{1}{n}\right)-\frac{1}{n}\right]^k\right\}.$$

(b) (5 marks) Find $E[X]$. Hint: label the red balls $1, \ldots, n$ and define

$$X_i = \begin{cases} 1 & \text{if the red ball labelled } i \text{ is in urn 1 after the } k\text{th stage} \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

(a) From the binomial theorem

$$\begin{aligned}
&\frac{1}{2}\left\{\left[\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]^k + \left[\left(1-\frac{1}{n}\right)-\frac{1}{n}\right]^k\right\} \\
&\quad= \frac{1}{2}\left\{\sum_{m=0}^{k}\binom{k}{m}\left(1-\frac{1}{n}\right)^{k-m}\left(\frac{1}{n}\right)^m + \sum_{m=0}^{k}\binom{k}{m}\left(1-\frac{1}{n}\right)^{k-m}\left(-\frac{1}{n}\right)^m\right\} \\
&\quad= \frac{1}{2}\left\{\sum_{m=0}^{k}\binom{k}{m}\left(1-\frac{1}{n}\right)^{k-m}\left[\left(\frac{1}{n}\right)^m + \left(-\frac{1}{n}\right)^m\right]\right\} \\
&\quad= \frac{1}{2}\left\{\sum_{\substack{m=0: \\ m \text{ even}}}^{k} 2\binom{k}{m}\left(\frac{1}{n}\right)^m\left(1-\frac{1}{n}\right)^{k-m}\right\} \\
&\quad= \sum_{\substack{m=0: \\ m \text{ even}}}^{k} \binom{k}{m}\left(\frac{1}{n}\right)^m\left(1-\frac{1}{n}\right)^{k-m}.
\end{aligned}$$

(b) With X_i as defined in the hint, i.e.,

$$X_i = \begin{cases} 1 & \text{if the red ball labelled } i \text{ is in urn 1 after the } k\text{th stage} \\ 0 & \text{otherwise,} \end{cases}$$

we have that the number of red balls in urn 1 after the kth stage is $X = X_1 + \ldots + X_n$. Therefore, $E[X] = E[X_1] + \ldots + E[X_n]$. By symmetry, $E[X_i]$ is the same for all i, so we have

$$E[X] = nE[X_1] = nP(\text{the red ball labelled 1 is in urn 1 after the } k\text{th stage}).$$

Now this red ball will be in urn 1 after the kth stage if and only if in the first k stages it was picked an even number of times. Since at each stage, and independently from stage to stage, this ball will be picked with probability $1/n$, the number of times this ball is picked in the first k stages has a binomial distribution with parameters k and $1/n$. Thus, the probability that it is picked an even number of times is

$$\begin{aligned} E[X_1] &= P(\text{red ball 1 is picked an even number of times during the first } k \text{ stages}) \\ &= \sum_{\substack{m=0: \\ m \text{ even}}}^{k} P(\text{red ball 1 is picked } m \text{ times during the first } k \text{ stages}) \\ &= \sum_{\substack{m=0: \\ m \text{ even}}}^{k} \binom{k}{m} \left(\frac{1}{n}\right)^m \left(1 - \frac{1}{n}\right)^{k-m} \\ &= \frac{1}{2}\left\{\left[\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]^k + \left[\left(1-\frac{1}{n}\right)-\frac{1}{n}\right]^k\right\} \quad \text{(from part(a))} \\ &= \frac{1}{2}\left[1 + \left(1-\frac{2}{n}\right)^k\right], \end{aligned}$$

and

$$E[X] = \frac{n}{2}\left[1 + \left(1-\frac{2}{n}\right)^k\right].$$

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2. (5 marks) Let $X_1, X_2, \ldots$ be a sequence of continuous, independent, and identically distributed random variables. Let

$$N = \min\{n : X_1 \geq X_2 \geq X_3 \geq \ldots \geq X_{n-1}, X_{n-1} < X_n\}.$$

Find $E[N]$.

Solution: The possible values of N are $\{2, 3, 4, \ldots\}$. For n in this set we have that $N \geq n$ if and only if $X_1 \geq X_2 \geq X_3 \geq \ldots \geq X_{n-1}$. Therefore,

$$P(N \geq n) = P(X_1 \geq X_2 \geq \ldots \geq X_{n-1}) = \frac{1}{(n-1)!},$$

since each ordering of $X_1, X_2, \ldots, X_{n-1}$ is equally likely. By Theorem 10.2,

$$\begin{aligned} E[N] &= \sum_{n=1}^{\infty} P(N \geq n) \\ &= 1 + \sum_{n=2}^{\infty} P(N \geq n) \\ &= 1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} = e. \end{aligned}$$

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3. (5 marks) Let X be a random variable and let $f(\cdot)$ and $g(\cdot)$ be nondecreasing bounded functions. In this problem we wish to show that $f(X)$ and $g(X)$ are positively correlated random variables. To do this let X' be a random variable independent of X and with the same distribution as X. First, show that

$$\mathrm{Cov}(f(X), g(X)) = \frac{1}{2}\mathrm{Cov}(f(X) - f(X'), g(X) - g(X')).$$

Deduce from the above equality that $\mathrm{Cov}(f(X), g(X)) \geq 0$.

Solution: From properties of covariance, we have

$$\begin{aligned}
&\mathrm{Cov}(f(X) - f(X'), g(X) - g(X')) \\
&\quad = \mathrm{Cov}(f(X), g(X)) - \mathrm{Cov}(f(X), g(X')) - \mathrm{Cov}(f(X'), g(X)) + \mathrm{Cov}(f(X'), g(X')).
\end{aligned}$$

The two middle terms on the right hand side above are equal to 0 since X and X' are independent, while the first and last terms are equal to one another since X and X' are identically distributed. Thus,

$$\mathrm{Cov}(f(X) - f(X'), g(X) - g(X')) = 2\mathrm{Cov}(f(X), g(X)).$$

Since both $f(X) - f(X')$ and $g(X) - g(X')$ are zero mean random variables, we have

$$\mathrm{Cov}(f(X), g(X)) = \frac{1}{2}E[(f(X) - f(X'))(g(X) - g(X'))].$$

But as both $f(\cdot)$ and $g(\cdot)$ are nondecreasing functions, $(f(X) - f(X'))(g(X) - g(X'))$ is nonnegative, since if $X > X'$ then $f(X) - f(X')$ and $g(X) - g(X')$ are both nonnegative while if $X < X'$ then $f(X) - f(X')$ and $g(X) - g(X')$ are both nonpositive (clearly if $X = X'$ then $f(X) - f(X')$ and $g(X) - g(X')$ are both equal to zero). So $\mathrm{Cov}(f(X), g(X)) \geq 0$.

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4. (5 marks) Let $(X_1, \ldots, X_k)^T$ have a Multinomial distribution with parameters n and $p_1, \ldots, p_k$. Find the correlation between X_i and X_j for any $i, j = 1, \ldots, k$. (cf. Problem 5 on Homework 5).

Solution: From Problem 5 of Homework 5, $EX_iX_j] = n(n-1)p_ip_j$ for $i \neq j$. The marginal distributions of X_i and X_j are Binomial(n, p_i) and Binomial(n, p_j), respectively. Therefore, $E[X_i] = np_i$, $\text{Var}(X_i) = np_i(1-p_i)$, $E[X_j] = np_j$, and $\text{Var}(X_j) = np_j(1-p_j)$. Then the covariance between X_i and X_j is

$$\text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j] = n(n-1)p_ip_j - np_inp_j = -np_ip_j$$

and the correlation between X_i and X_j is

$$\rho(X_i, X_j) = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i)\text{Var}(X_j)}} = \frac{-np_ip_j}{\sqrt{np_i(1-p_i)np_j(1-p_j)}} = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}.$$

If $i = j$ then $\rho(X_i, X_i) = 1$ for $i = 1, \ldots, n$.

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5. (6 marks) Let $X_1, \dots, X_n$ be independent $U(0,1)$ random variables and let $X_{(1)} = \min(X_1, \dots, X_n)$ and $X_{(n)} = \max(X_1, \dots, X_n)$ denote the first and last order statistics. Find $\rho(X_{(1)}, X_{(n)})$, the correlation coefficient between $X_{(1)}$ and $X_{(n)}$.

Solution: To compute $\rho(X_{(1)}, X_{(n)})$ we will need $\text{Cov}(X_{(1)}, X_{(n)})$, $\text{Var}(X_{(1)})$ and $\text{Var}(X_{(n)})$. The pdf of $X_{(1)}$ is $f_1(x_1) = n(1-x_1)^{n-1}I_{(0,1)}(x_1)$, which is the pdf of a Beta$(1,n)$ distribution. The pdf of $X_{(n)}$ is $f_n(x_n) = nx^{n-1}I_{(0,1)}(x_n)$, which is the pdf of a Beta$(n,1)$ distribution. Therefore,

$$\begin{aligned} E[X_{(1)}] &= \frac{1}{n+1} \qquad \text{Var}(X_{(1)}) = \frac{n}{(n+2)(n+1)^2} \\ E[X_{(n)}] &= \frac{n}{n+1} \qquad \text{Var}(X_{(n)}) = \frac{n}{(n+2)(n+1)^2}. \end{aligned}$$

We still need to compute $E[X_{(1)}X_{(n)}]$ to get $\text{Cov}(X_{(1)}, X_{(n)})$. The joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$f_{1n}(x_1, x_n) = \begin{cases} n(n-1)(x_n - x_1)^{n-2} & \text{for } 0 < x_1 < x_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} E[X_{(1)}X_{(n)}] &= \int_0^1 \int_0^{x_n} x_1 x_n n(n-1)(x_n - x_1)^{n-2} dx_1 dx_n \\ &= \int_0^1 x_n n(n-1) \int_0^1 y x_n (x_n - y x_n)^{n-2} x_n dy dx_n \\ &\qquad \text{(by the substitution } x_1 = y x_n) \\ &= \int_0^1 n(n-1) x_n^{n+1} \Big[\int_0^1 y(1-y)^{n-2} dy \Big] dx_n \\ &= \int_0^1 n(n-1) x_n^{n+1} \frac{\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)} dx_n \\ &\qquad \text{(since the inner integral is proportional to a Beta}(2, n-1) \text{ pdf)} \\ &= \int_0^1 x_n^{n+1} dx_n = \frac{x_n^{n+2}}{n+2}\bigg|_0^1 = \frac{1}{n+2}. \end{aligned}$$

Then

$$\text{Cov}(X_{(1)}, X_{(n)}) = E[X_{(1)}X_{(n)}] - E[X_{(1)}]E[X_{(n)}] = \frac{1}{n+2} - \frac{n}{(n+1)^2} = \frac{1}{(n+2)(n+1)^2}$$

and

$$\rho(X_{(1)}, X_{(n)}) = \frac{\text{Cov}(X_{(1)}, X_{(n)})}{\sqrt{\text{Var}(X_{(1)})\text{Var}(X_{(n)})}} = \frac{1/((n+2)(n+1)^2)}{n/((n+2)(n+1)^2)} = \frac{1}{n}.$$