Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 7   Due March 28, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 30

Student Number Name

1. (5 marks) Let X_1 and X_2 be random variables and Y a random vector. Use the Law of Total Expectation to show the following generalization of the conditional variance formula:

$$\mathrm{Cov}(X_1, X_2) = E[\mathrm{Cov}((X_1, X_2) \mid Y)] + \mathrm{Cov}(E[X_1 \mid Y], E[X_2 \mid Y]).$$

Here, $\mathrm{Cov}((X_1, X_2) \mid Y) = E[X_1 X_2 \mid Y] - E[X_1 \mid Y] E[X_2 \mid Y]$ is that function of Y whose value when $Y = y$ is the covariance between X_1 and X_2 with respect to the conditional joint distribution of X_1 and X_2 given $Y = y$.

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2. (6 marks) A coin, having probability p of coming up heads, is successively flipped until at least one head and one tail have been flipped.

(a) (3 marks) Find the expected number of flips needed.

(b) (3 marks) Find the expected number of flips that land on heads.

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3. (7 marks) Let X_i, $i \geq 1$, be independent Uniform(0,1) random variables, and let $X_0 = x$, for some $x \in [0, 1]$. Define $N(x)$ by

$$N(x) = \min(n \geq 1 : X_n < X_{n-1}).$$

In words, $N(x)$ is the first "time" that an X value is smaller than the immediately preceding X value. Let $\overline{N}(x) = E[N(x)]$, for $0 \leq x \leq 1$.

(a) (4 marks) Derive an integral equation for $\overline{N}(x)$ by conditioning on X_1.

(b) (3 marks) Differentiate both sides of the equation derived in part(a) and solve the resulting equation obtained (note that there is a boundary condition $\overline{N}(1) = 1$).

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4. (6 marks) A fair coin is tossed successively. Let K_n be the number of tosses until n consecutive heads occur for the first time. By conditioning on K_{n-1} express $E[K_n]$ in terms of $E[K_{n-1}]$, then solve this recursion (note that $E[K_1] = 2$) and find $E[K_n]$.

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5. (6 marks) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{y^3}{2} e^{-y(x+1)} & \text{for } x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathrm{Var}(X)$ using the conditional variance formula.