For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:

(1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.
(2) Write your whole solution using your own paper, and make sure to number your solution.
(3) Write your solution using document creation software (e.g., Word or LaTeX).

Write your name and student number on the first page of each solution.

For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.
1. Suppose $X$ is a random variable whose moments, for $n \geq 1$, are given by $E[X^n] = n!/\lambda^n$, where $\lambda > 0$. Find the distribution of $X$. *Hint:* Write the moment generating function of $X$ as a Taylor series expansion; assume you can take the expectation inside the infinite sum.
2. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, each with the same distribution as the random variable $X$, and let $N$ be a random variable on the nonnegative integers independent of the $X_i$’s. Let $m_X(t)$ denote the moment generating function of $X$ and let $m_N(t)$ denote the moment generating function of $N$. Finally, let $Y$ denote the random sum

$$Y = \sum_{i=1}^{N} X_i$$

if $N \geq 1$ and $Y = 0$ if $N = 0$.

(a) By conditioning on $N$, show that the moment generating function of $Y$ is given by

$$m_Y(t) = m_N(\ln(m_X(t))).$$

(b) Let $N$ have a Poisson($\lambda$) distribution and suppose $N$ independent Bernoulli trials are conducted, where the probability of success in each trial is $p$. Let $Y$ denote the total number of successes in the conducted trials. Compute the moment generating function of $Y$ and use this to determine the distribution of $Y$. 
3.  (a) Suppose $X$ is a nonnegative random variable with $P(X \geq 10) = 0.1$. What is the smallest possible value of $E[X]$? Give a distribution for $X$ under which the mean of $X$ achieves this smallest value.

(b) We have shown that if $X$ is a random variable with mean $\mu$ and variance $\sigma^2$, where $\sigma^2 < \infty$, then $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ for any positive integer $k$, using Chebyshev’s inequality. For any positive integer $k$, give a distribution for $X$ that satisfies $P(|X - \mu| \geq k\sigma) = 1/k^2$. 
4. **Lower Bound.** Let $X$ be a nonnegative random variable bounded by a fixed positive constant $M$, i.e., $P(0 \leq X \leq M) = 1$. Show that

$$P(X \geq a) \geq \frac{E[X] - a}{M - a}$$

for $a \in [0, M)$. **Hint:** Show that $X \leq MI_A + aI_{A^c}$, where $A = \{X \geq a\}$ and $I_B$ is the indicator of the event $B$, and take expectations.
5. Consider the following generalization of the weak law of large numbers. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma_i^2$ (thus, the $X$‘s are not necessarily identically distributed). Let $X_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ denote the sample mean of $X_1, \ldots, X_n$, for $n \geq 1$. Show that if there exists an $M > 0$ such that $\sigma_i^2 \leq M$ for all $i \geq 1$, then the weak law of large numbers holds; i.e., show that for any $\epsilon > 0$,

$$P(|X_n - \mu| > \epsilon) \to 0 \quad \text{as } n \to \infty$$