Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 7 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 30

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1. (5 marks) Let X_1 and X_2 be random variables and Y a random vector. Use the Law of Total Expectation to show the following generalization of the conditional variance formula:

$$\mathrm{Cov}(X_1, X_2) = E[\mathrm{Cov}((X_1, X_2) \mid Y)] + \mathrm{Cov}(E[X_1 \mid Y], E[X_2 \mid Y]).$$

Here, $\mathrm{Cov}((X_1, X_2) \mid Y) = E[X_1 X_2 \mid Y] - E[X_1 \mid Y]E[X_2 \mid Y]$ is that function of Y whose value when $Y = y$ is the covariance between X_1 and X_2 with respect to the conditional joint distribution of X_1 and X_2 given $Y = y$.

Solution: By the Law of Total Expectation

$$\begin{aligned} E[\mathrm{Cov}((X_1, X_2) \mid Y)] &= E[E[X_1 X_2 \mid Y]] - E[E[X_1 \mid Y]E[X_2 \mid Y]] \\ &= E[X_1 X_2] - E[E[X_1 \mid Y]E[X_2 \mid Y]] \qquad (1) \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{Cov}(E[X_1 \mid Y], E[X_2 \mid Y]) &= E[E[X_1 \mid Y]E[X_2 \mid Y]] - E[E[X_1 \mid Y]]E[E[X_2 \mid Y]] \\ &= E[E[X_1 \mid Y]E[X_2 \mid Y]] - E[X_1]E[X_2] \qquad (2) \end{aligned}$$

again by the Law of Total Expectation. Summing up (1) and (2) gives

$$\begin{aligned} & E\left[\mathrm{Cov}(X_1, X_2) \mid Y\right] + \mathrm{Cov}(E[X_1 \mid Y], E[X_2 \mid Y]) \\ = \ & E[X_1 X_2] - E[E[X_1 \mid Y]E[X_2 \mid Y]] + E[E[X_1 \mid Y]E[X_2 \mid Y]] - E[X_1]E[X_2] \\ = \ & E[X_1 X_2] - E[X_1]E[X_2] \\ = \ & \mathrm{Cov}(X_1, X_2). \end{aligned}$$

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2. (6 marks) A coin, having probability p of coming up heads, is successively flipped until at least one head and one tail have been flipped.

(a) (3 marks) Find the expected number of flips needed.

(b) (3 marks) Find the expected number of flips that land on heads.

Solution:

(a) Let X denote the number of flips required until at least one head and one tail have been flipped and let Y indicate the outcome of the first flip:

$$Y = \begin{cases} 1 & \text{if the first flip is heads} \\ 0 & \text{if the first flip is tails.} \end{cases}$$

Then we have $E[X \mid Y = 1] = 1 + E[W_T]$ and $E[X \mid Y = 0] = 1 + E[W_H]$, where W_T and W_H are, respectively, the number of flips required, starting from the second flip, until a tails or a heads is flipped. Clearly, W_T has a Geometric$(1-p)$ distribution and W_H has a Geometric(p) distribution, so $E[W_T] = \frac{1}{1-p}$ and $E[W_H] = \frac{1}{p}$. By the Law of Total Expectation we have

$$E[X] = E[X \mid Y = 1]P(Y = 1) + E[X \mid Y = 0]P(Y = 0) = 1 + \frac{p}{1-p} + \frac{1-p}{p}.$$

(b) Let Z denote the number of flips that land on heads and let Y again denote the outcome of the first flip. Then conditioning on Y we have

$$E[Z] = E[Z \mid Y = 1]p + E[Z \mid Y = 0](1-p).$$

If the first flip is heads then the expected number of heads will be 1 plus $E[Z_1 - 1]$, where Z_1 is the number of flips starting from the second flip required to first flip a tails. This is Geometric$(1-p)$ with mean $\frac{1}{1-p}$. Therefore, $E[Z \mid Y = 1] = 1 + \frac{1}{1-p} - 1 = \frac{1}{1-p}$. If the first flip is tails then the total number of heads flipped will be just 1 (the last flip) so that $E[Z \mid Y = 0] = 1$. Putting this together we have

$$E[Z] = \frac{p}{1-p} + (1)(1-p) = \frac{(1-p)^2 + p}{1-p} = \frac{1-p+p^2}{1-p} = 1 + \frac{p^2}{1-p}.$$

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3. (7 marks) Let X_i, $i \geq 1$, be independent Uniform(0,1) random variables, and let $X_0 = x$, for some $x \in [0, 1]$. Define $N(x)$ by

$$N(x) = \min(n \geq 1 : X_n < X_{n-1}).$$

In words, $N(x)$ is the first "time" that an X value is smaller than the immediately preceding X value. Let $\overline{N}(x) = E[N(x)]$, for $0 \leq x \leq 1$.

(a) (4 marks) Derive an integral equation for $\overline{N}(x)$ by conditioning on X_1.

(b) (3 marks) Differentiate both sides of the equation derived in part(a) and solve the resulting equation obtained (note that there is a boundary condition $\overline{N}(1) = 1$).

Solution:

(a) Conditioning on X_1, we have

$$\overline{N}(x) = E[N(x)] = \int_0^1 E[N(x) \mid X_1 = t] f_{X_1}(t) dt = \int_0^1 E[N(x) \mid X_1 = t] dt,$$

since $f_{X_1}(t) = 1$ for $0 \leq t \leq 1$ for a U(0,1) distribution. For $E[N(x) \mid X_1 = t]$, if $0 \leq t < x$ then $N(x) = 1$ (because at time $n = 1$ we got a decrease). If $x \leq t \leq 1$, then $N(x)$ is equal to 1 (for X_1) plus the additional number of X's we need to observe until we see a decrease. But this latter number has the same distribution as $N(t)$ (it's as if we started with $X_0 = t$). To summarize,

$$E[N(x) \mid X_1 = t] = \begin{cases} 1 & \text{if } 0 \leq t < x \\ 1 + E[N(t)] & \text{if } x \leq t \leq 1. \end{cases}$$

Plugging this back into the integral equation, we get

$$\overline{N}(x) = \int_0^x (1) dt + \int_x^1 (1 + \overline{N}(t)) dt = 1 + \int_x^1 \overline{N}(t) dt$$

(b) Differentiating (with respect to x) the integral equation obtained in part(a), we obtain $\overline{N}'(x) = -\overline{N}(x)$. We also have a boundary condition $\overline{N}(1) = 1$ because $N(1) = 1$ with probability 1 (because X_1 will be less than 1 with probability 1). The solution to the differential equation from part(b) is easily seen to be $\overline{N}(x) = ce^{-x}$, for some constant c. The boundary condition $\overline{N}(1) = 1$ gives $ce^{-1} = 1$, which implies $c = e$. Thus, the final answer is

$$\overline{N}(x) = e^{1-x} \quad \text{for } 0 \leq x \leq 1.$$

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4. (6 marks) A fair coin is tossed successively. Let K_n be the number of tosses until n consecutive heads occur for the first time. By conditioning on K_{n-1} express $E[K_n]$ in terms of $E[K_{n-1}]$, then solve this recursion (note that $E[K_1] = 2$) and find $E[K_n]$.

Solution: Suppose that $K_{n-1} = i$ is given. Then the first time that $n-1$ consecutive heads occurred was on toss i. Therefore, if toss $i+1$ is also heads then n consecutive heads occurred for the first time on toss $i+1$ and we have $K_n = i$. If toss $i+1$ is tails, then we have to start over and the additional number of tosses we need to get n consecutive heads has the same distribution as K_n. Thus,

$$E[K_n \mid K_{n-1} = i] = (i+1)\frac{1}{2} + (i+1+E[K_n])\frac{1}{2},$$

which gives

$$\begin{aligned} E[K_n \mid K_{n-1}] &= (K_{n-1}+1)\frac{1}{2} + (K_{n-1}+1+E[K_n])\frac{1}{2} \\ &= K_{n-1}+1+\frac{1}{2}E[K_n]. \end{aligned}$$

Taking expectation and using the Law of Total Expectation, we have

$$E[K_n] = E[E[K_n \mid K_{n-1}]] = E[K_{n-1}] + 1 + \frac{1}{2}E[K_n]$$

or

$$E[K_n] = 2 + 2E[K_{n-1}].$$

We can now solve the recursion. We have the initial condition that $E[K_1] = 2$ (since K_1 has a geometric(1/2) distribution). Then,

$$\begin{aligned} E[K_n] &= 2 + 2E[K_{n-1}] \\ &= 2 + 2(2 + 2E[K_{n-2}]) \\ &= 2 + 2^2 + 2^2E[K_{n-2}] \\ &= 2 + 2^2 + 2^3 + 2^3E[K_{n-3}] \\ &\vdots \\ &= 2 + 2^2 + 2^3 + \ldots + 2^{n-1} + 2^{n-1}E[K_1] \\ &= 2 + 2^2 + 2^3 + \ldots + 2^{n-1} + 2^n = 2\sum_{k=0}^{n-1} 2^k = 2\frac{1-2^n}{1-2} = 2(2^n - 1). \end{aligned}$$

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5. (6 marks) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{y^3}{2}e^{-y(x+1)} & \text{for } x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathrm{Var}(X)$ using the conditional variance formula.

Solution: To find $\mathrm{Var}(X)$ using the conditional variance formula, we need the mean and variance of the conditional distribution of X given $Y = y$. The marginal pdf of Y is

$$f_Y(y) = \int_0^\infty \frac{y^3}{2}e^{-y(x+1)}dx = \frac{y^3}{2}e^{-y}\int_0^\infty e^{-yx}dx = \frac{y^3}{2}e^{-y}\frac{1}{y} = \frac{y^2}{2}e^{-y}$$

for $y \geq 0$ and $f_Y(y) = 0$ for $y < 0$ (that is, the marginal distribution of Y is $\mathrm{Gamma}(3,1)$). Then the conditional pdf of X given $Y = y$ is

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{(y^3/2)e^{-y(x+1)}}{(y^2/2)e^{-y}} = ye^{-yx},$$

for $x \geq 0$, and $f(x \mid y) = 0$ for $x < 0$. That is, the conditional distribution of X given $Y = y$ is Exponential (y). The mean and variance of this conditional distribution are $E[X \mid Y = y] = \frac{1}{y}$ and $\mathrm{Var}(X \mid Y = y) = \frac{1}{y^2}$, respectively. Thus we have

$$E[X \mid Y] = \frac{1}{Y} \quad \text{and} \quad \mathrm{Var}(X \mid Y) = \frac{1}{Y^2}.$$

The conditional variance formula is $\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$, where the mean and variance are computed with respect to the marginal distribution of Y. Thus, we have

$$\mathrm{Var}(X) = \mathrm{Var}\left(\frac{1}{Y}\right) + E\left[\frac{1}{Y^2}\right] = E\left[\frac{1}{Y^2}\right] - E\left[\frac{1}{Y}\right]^2 + E\left[\frac{1}{Y^2}\right] = 2E\left[\frac{1}{Y^2}\right] - E\left[\frac{1}{Y}\right]^2.$$

As $Y \sim \mathrm{Gamma}(3,1)$, we have

$$\mathrm{Var}(X) = 2\int_0^\infty \frac{1}{y^2}\frac{y^2}{2}e^{-y}dy - \left[\int_0^\infty \frac{1}{y}\frac{y^2}{2}e^{-y}dy\right]^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$