Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 8 Due April 4, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 25

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1. (4 marks) For each of the following parts, either give a distribution such that the given function is the mgf of that distribution, or show why the given function cannot be the mgf of any distribution.

(a) (2 marks) $M(t) = \frac{1}{4}(1 + e^t)^2$.

(b) (2 marks) $M(t) = 1 + t^2$. *Hint:* Consider $\mathrm{Var}(X^2)$.

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2. (6 marks) Let $X_1, X_2, \ldots$ be a sequence of random variables such that X_n has a Binomial distribution with parameters n and p_n. Assume that the sequence $\{p_n\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} np_n = \lambda$, where $\lambda > 0$. Let X have a Poisson distribution with parameter λ. Let $M_{X_n}(t)$ be the mgf of X_n and let $M_X(t)$ be the mgf of X. Compute $M_{X_n}(t)$ and $M_X(t)$ and show that $M_{X_n}(t) \to M_X(t)$ as $n \to \infty$ for every t. You may use the fact that if $\{x_n\}$ is a sequence satisfying $x_n \to x$ as $n \to \infty$ then $(1 + \frac{x_n}{n})^n \to e^x$ as $n \to \infty$.

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3. (3 marks) Let $\mu > 0$ be given. Let k be a positive integer. Give an example of a distribution with mean μ and finite variance such that if X is a random variable with that distribution then

$$P(|X - E[X]| \geq k\sqrt{\mathrm{Var}(X)}) = \frac{1}{k^2}.$$

(i.e., Chebyshev's inequality is achieved).

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4. (6 marks) Let X be a random variable with mgf $M_X(t)$, and suppose that $M_X(t)$ exists for all t.

(a) (3 marks) Using Markov's inequality, show that for any $t > 0$ and any $a \in \mathbb{R}$,

$$P(X \geq a) \leq e^{-at} M_X(t).$$

(b) (3 marks) For any fixed a, minimizing the right hand side of the inequality in part(a) over $t > 0$ gives what is called the *Chernoff bound* to $P(X \geq a)$. If $X \sim N(0, 1)$, find Chernoff's bound to $P(X \geq a)$ for $a > 0$.

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5. (6 marks) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ denote the sample mean of $X_1, \ldots, X_n$ for $n \geq 1$. Suppose that $\frac{1}{n}\sum_{i=1}^n \mu_i \to \mu$ as $n \to \infty$, for some $\mu \in \mathbb{R}$. Finally, suppose that $\sigma_i^2 \leq M$ for all $i \geq 1$, for some finite, positive M. Show that for any $\epsilon > 0$, $P(|\overline{X}_n - \mu| \geq \epsilon) \to 0$ as $n \to \infty$.