Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Homework 8 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:

  (1) Use your own paper.

  (2) Use a tablet, such as an ipad.

  (3) Use document creation software, such as Word or LaTeX.

- Write your name and student number on the first page of each solution, and number your solution.

- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 25

Student Number Name

1. (4 marks) For each of the following parts, either give a distribution such that the given function is the mgf of that distribution, or show why the given function cannot be the mgf of any distribution.

(a) (2 marks) $M(t) = \frac{1}{4}(1 + e^t)^2$.

(b) (2 marks) $M(t) = 1 + t^2$. *Hint:* Consider $\mathrm{Var}(X^2)$.

Solution:

(a) We have $M(t) = \frac{1}{4}(1 + 2e^t + e^{2t})$. By inspection we can see that if X is a discrete random variable on $\{0, 1, 2\}$ with probabilities $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, and $P(X = 2) = \frac{1}{4}$, then X will have mgf $M(t)$.

(b) The function $M(t) = 1 + t^2$ cannot be the mgf of any distribution. If it were the mgf of some random variable X, then we would have $E[X] = M'(0) = 0, E[X^2] = M''(0) = 2$, and $E[X^n] = M^{(n)}(0) = 0$. These moments would imply that $\mathrm{Var}(X^2) = E[X^4] - E[X^2]^2 = 0 - 2^2 = -4$, which is impossible for a variance. Therefore, $M(t)$ cannot be the mgf of any distribution.

Student Number Name

2. (6 marks) Let $X_1, X_2, \ldots$ be a sequence of random variables such that X_n has a Binomial distribution with parameters n and p_n. Assume that the sequence $\{p_n\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} np_n = \lambda$, where $\lambda > 0$. Let X have a Poisson distribution with parameter λ. Let $M_{X_n}(t)$ be the mgf of X_n and let $M_X(t)$ be the mgf of X. Compute $M_{X_n}(t)$ and $M_X(t)$ and show that $M_{X_n}(t) \to M_X(t)$ as $n \to \infty$ for every t. You may use the fact that if $\{x_n\}$ is a sequence satisfying $x_n \to x$ as $n \to \infty$ then $(1 + \frac{x_n}{n})^n \to e^x$ as $n \to \infty$.

Solution: If we write $X_n = Y_1 + \ldots + Y_n$, where $Y_1, \ldots, Y_n$ are i.i.d. Bernoulli(p_n) random variables, then the mgf of X_n is given by $M_{X_n}(t) = (M_Y(t))^n$, where $M_Y(t)$ is the common mgf of each of the Y_i's. Since Y_i has a Bernoulli(p_n) distribution, we have $M_Y(t) = e^{(0)t}(1 - p_n) + e^{(1)t}p_n = 1 - p_n + p_n e^t$. Thus, the mgf of X_n is given by

$$M_{X_n}(t) = (p_n e^t + 1 - p_n)^n = (1 - p_n(1 - e^t))^n = \left(1 - \frac{np_n(1 - e^t)}{n}\right)^n. \tag{1}$$

For the Poisson(λ) distribution we can directly compute the mgf as

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}. \tag{2}$$

In (1), as $n \to \infty$ we have that $np_n(1 - e^t) \to \lambda(1 - e^t)$ for every t, and we obtain that

$$\left(1 - \frac{np_n(1 - e^t)}{n}\right)^n \to e^{-\lambda(1-e^t)} = e^{\lambda(e^t - 1)}.$$

as $n \to \infty$. The function on the right hand side above is the mgf of the Poisson(λ) distribution, as given in (2).

Student Number Name

3. (3 marks) Let $\mu > 0$ be given. Let k be a positive integer. Give an example of a distribution with mean μ and finite variance such that if X is a random variable with that distribution then

$$P(|X - E[X]| \geq k\sqrt{\mathrm{Var}(X)}) = \frac{1}{k^2}.$$

(i.e., Chebyshev's inequality is achieved).

Solution: Let $k \geq 1$ be a given positive integer. Let Y have a discrete distribution with the following pmf:

$$f(y) = \begin{cases} \frac{1}{2k^2} & \text{if } y = -k\sigma \\ 1 - \frac{1}{k^2} & \text{if } y = 0 \\ \frac{1}{2k^2} & \text{if } y = k\sigma, \end{cases}$$

where $\sigma > 0$ is arbitrary, and set $X = Y + \mu$. We first verify that the variance of X is σ^2. Note that $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Since the distribution of Y is symmetric about 0, the mean of Y is 0, and so

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = E[Y^2] = k^2\sigma^2\left(\frac{1}{2k^2} + \frac{1}{2k^2}\right) = \sigma^2.$$

On the other hand,

$$P(|X - \mu| \geq k\sigma) = P(Y = -k\sigma) + P(Y = k\sigma) = \frac{1}{2k^2} + \frac{1}{2k^2} = \frac{1}{k^2}.$$

Student Number Name

4. (6 marks) Let X be a random variable with mgf $M_X(t)$, and suppose that $M_X(t)$ exists for all t.

(a) (3 marks) Using Markov's inequality, show that for any $t > 0$ and any $a \in \mathbb{R}$,

$$P(X \geq a) \leq e^{-at} M_X(t).$$

(b) (3 marks) For any fixed a, minimizing the right hand side of the inequality in part(a) over $t > 0$ gives what is called the *Chernoff bound* to $P(X \geq a)$. If $X \sim N(0,1)$, find Chernoff's bound to $P(X \geq a)$ for $a > 0$.

Solution:

(a) Let $t > 0$ and $a \in \mathbb{R}$ be given. Then

$$P(X \geq a) = P(tX \geq ta) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-at} M_X(t),$$

where the inequality above is from Markov's inequality.

(b) Let $a > 0$ be given. If $X \sim N(0,1)$, then $M_X(t) = e^{t^2/2}$, and Chernoff's bound is

$$P(X \geq a) \leq \min_{t>0} e^{-at} e^{t^2/2} = \min_{t>0} e^{(t^2-2at)/2}, \tag{3}$$

which is minimized over $t > 0$ by minimizing $t^2 - 2at$ over $t > 0$ since the exponential function is strictly increasing. Differentiating $t^2 - 2at$ with respect to t and setting this to 0 gives $2t - 2a = 0$ or $t = a$, which is positive. Thus, the minimum over $t > 0$ is obtained at $t = a$ and plugging this into the right hand side of (3) gives

$$P(X \geq a) \leq e^{(a^2-2a^2)/2} = e^{-a^2/2}.$$

Student Number Name

5. (6 marks) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ denote the sample mean of $X_1, \ldots, X_n$ for $n \geq 1$. Suppose that $\frac{1}{n}\sum_{i=1}^n \mu_i \to \mu$ as $n \to \infty$, for some $\mu \in \mathbb{R}$. Finally, suppose that $\sigma_i^2 \leq M$ for all $i \geq 1$, for some finite, positive M. Show that for any $\epsilon > 0$, $P(|\overline{X}_n - \mu| \geq \epsilon) \to 0$ as $n \to \infty$.

Solution: Let $\epsilon > 0$ be given. First note that $E[\overline{X}_n] = \frac{1}{n}\sum_{i=1}^n \mu_i$ and $\mathrm{Var}(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^n \sigma_i^2$. For ease of notation, let $\overline{\mu}_n = \frac{1}{n}\sum_{i=1}^n \mu_i$. Using the triangle inequality we have

$$\begin{aligned} P(|\overline{X}_n - \mu| \geq \epsilon) &= P(|\overline{X}_n - \overline{\mu}_n + \overline{\mu}_n - \mu| \geq \epsilon) \\ &\leq P(|\overline{X}_n - \overline{\mu}_n| + |\overline{\mu}_n - \mu| \geq \epsilon) \end{aligned}$$

Since $\overline{\mu}_n \to \mu$ we can choose N such that $|\overline{\mu}_n - \mu| < \epsilon$ for all $n \geq N$. Then for such n, since $E[\overline{X}_n] = \overline{\mu}_n$, and using Chebyshev's inequality, we have

$$\begin{aligned} P(|\overline{X}_n - \mu| \geq \epsilon) &\leq P(|\overline{X}_n - \overline{\mu}_n| \geq \epsilon - |\overline{\mu}_n - \mu|) \\ &\leq \frac{\mathrm{Var}(\overline{X}_n)}{(\epsilon - |\overline{\mu}_n - \mu|)^2} \\ &= \frac{1}{n^2(\epsilon - |\overline{\mu}_n - \mu|)^2}\sum_{i=1}^n \sigma_i^2. \end{aligned}$$

If there is an M such that $\sigma_i^2 \leq M$ for all $i \geq 1$, then

$$\sum_{i=1}^n \sigma_i^2 \leq \sum_{i=1}^n M = Mn$$

and so

$$P(|\overline{X}_n - \mu| \geq \epsilon) \leq \frac{Mn}{n^2(\epsilon - |\overline{\mu}_n - \mu|)^2} = \frac{M}{n(\epsilon - |\overline{\mu}_n - \mu|)^2} \to 0 \quad \text{as } n \to \infty.$$

Thus, $\overline{X}_n$ converges to μ in probability as $n \to \infty$.