• For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:

(1) Start your solution in the space provided right after the problem statement, and use your own paper if you need extra pages.

(2) Write your whole solution using your own paper, and make sure to number your solution.

(3) Write your solution using document creation software (e.g., Word or LaTeX).

• Write your name and student number on the first page of each solution.

• For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Total Marks 27
1. Let $X_1, X_2, \ldots$ be a sequence of random variables that converges in probability to a random variable $X$. Also, suppose there is some finite $M$ such that $P(|X_n| \leq M) = 1$ for every $n$.

(a) Show that $P(|X| \leq M) = 1$.

(b) Show that $\lim_{n \to \infty} E[|X_n - X|] = 0$.

Solutions: (8 marks)

(a) (3 marks) Let $\epsilon > 0$. We have that

$$P(|X| > M + \epsilon) = P(|X - X_n + X_n| > M + \epsilon)$$

$$\leq P(|X - X_n| + |X_n| > M + \epsilon) \quad \text{(triangle inequality)}$$

$$= P(|X - X_n| > M - |X_n| + \epsilon)$$

$$\leq P(|X - X_n| > \epsilon) \quad (M - |X_n| \geq 0 \text{ with prob. 1})$$

$$\to 0 \quad \text{as } n \to \infty$$

since $X_n \to X$ in probability by assumption. Therefore, $P(|X| > M + \epsilon) = 0$ for every $\epsilon > 0$. But this can only be true if $P(|X| > M) = 0$ (i.e., $P(|X| \leq M) = 1$).

(b) (5 marks) Let $B = \{\omega : |X(\omega)| \leq M\}$ and $B_n = \{\omega : |X_n(\omega)| \leq M\}$. By part(a), $P(B) = 1$ and $P(B_n) = 1$ by assumption. If $\omega \in A_n(\epsilon)$ then $|X_n(\omega) - X(\omega)| \leq \epsilon$ while if $\omega \in B \cap B_n$ then $|X_n(\omega) - X(\omega)| \leq 2M$. Therefore, we have $|X_n(\omega) - X(\omega)|^r \leq \epsilon^r I_{A_n(\epsilon)}(\omega) + (2M)^r I_{A_n(\epsilon)}(\omega)$ for all $\omega \in B \cap B_n$. Since $P(B \cap B_n) = 1$ then

$$|X_n - X|^r \leq \epsilon^r I_{A_n(\epsilon)} + (2M)^r I_{A_n(\epsilon)}$$

with probability 1. Taking expectations, we have

$$E[|X_n - X|^r] \leq \epsilon^r P(A_n(\epsilon)^c) + (2M)^r P(A_n(\epsilon))$$

$$\leq \epsilon^r + (2M)^r P(A_n(\epsilon)),$$

since $P(A_n(\epsilon)^c) \leq 1$. Since $X_n \to X$ in probability by assumption, we have that $P(A_n(\epsilon)) \to 0$ as $n \to \infty$, and so

$$\lim_{n \to \infty} E[|X_n - X|^r] \leq \epsilon^r,$$

and this is true for every $\epsilon > 0$. Thus, we must have $\lim_{n \to \infty} E[|X_n - X|^r] = 0$ (i.e. $X_n \to X$ in the $r$th mean as $n \to \infty$).
2. (a) Let $X_1, X_2, \ldots$ be a sequence of independent Exponential($\lambda$) random variables and let $\bar{X}_n = (X_1 + \ldots + X_n)/n$ denote the sample mean of the first $n$ random variables in the sequence. Let $Y_n = 1/\bar{X}_n$. Prove that $Y_n \to \lambda$ in probability.

(b) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2 < \infty$. Show that $\bar{X}_n$ converges to $\mu$ in mean square, where $\bar{X}_n$ is the sample mean of the first $n$ random variables in the sequence.

Solution: (6 marks)

(a) (4 marks) By the strong law of large numbers, $\bar{X}_n$ converges to $E[X_1] = 1/\lambda$ almost surely. Letting $g(x) = 1/x$ we have that $g(\cdot)$ is a continuous function on $(0, \infty)$. Therefore, from class we have seen that $g(\bar{X}_n)$ converges to $g(E[X_1])$ almost surely. But $g(\bar{X}_n) = Y_n$ and $g(E[X_1]) = \lambda$. Therefore, we conclude that $Y_n$ converges to $\lambda$ almost surely. Since almost sure convergence implies convergence in probability we also have that $Y_n$ converges to $\lambda$ in probability.

(b) (2 marks) We need to show that $E[(\bar{X}_n - \mu)^2] \to 0$ as $n \to \infty$. But

$$E[(\bar{X}_n - \mu)^2] = \text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$$

which clearly goes to 0 as $n \to \infty$. Therefore, $\bar{X}_n$ converges to $\mu$ in mean square.
3. (a) Show that if \( X_n \to X \) a.s. and \( Y_n \to Y \) a.s., then \( X_n + Y_n \to X + Y \) a.s.

(b) Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables, with mean \( \mu \) and variance \( \sigma^2 \). Let \( \bar{X}_n \) denote the sample mean of \( X_1, \ldots, X_n \), for \( n \geq 1 \). Show that

\[
\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \to \sigma^2 \quad \text{a.s. as } n \to \infty
\]

Hint: Expand the square.

**Solution: (8 marks)**

(a) (3 marks) Let \( A_1 = \{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \} \), \( A_2 = \{ \omega \in \Omega : \lim_{n \to \infty} Y_n(\omega) = Y(\omega) \} \), and \( A = A_1 \cap A_2 \). Then for \( \omega \in A \), we have that

\[
\lim_{n \to \infty} (X_n(\omega) + Y_n(\omega)) = \lim_{n \to \infty} X_n(\omega) + \lim_{n \to \infty} Y_n(\omega) = X(\omega) + Y(\omega).
\]

Since \( P(A_1) = 1 \) and \( P(A_2) = 1 \) by assumption, then \( P(A) = 1 \). Therefore, \( X_n + Y_n \to X + Y \) almost surely.

(b) (5 marks) Following the hint, we first write \( \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \), which is easily verified by expanding out the square and carrying the summation through. The sequence \( X_1^2, X_2^2, X_3^2, \ldots \) is a sequence of independent and identically distributed random variables. By the strong law of large numbers

\[
\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to E[X_1^2] \quad \text{with probability 1 as } n \to \infty,
\]

where

\[
E[X_1^2] = \text{Var}(X_1) + E[X_1]^2 = \sigma^2 + \mu^2.
\]

Again by the strong law of large numbers, \( \bar{X}_n \to \mu \) with probability 1 as \( n \to \infty \). From class notes we also have that

\[
-\bar{X}_n^2 \to -\mu^2 \quad \text{with probability 1 as } n \to \infty,
\]

since the function \( g(x) = -x^2 \) is a continuous function. From part(a) we have that

\[
\frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \to (\sigma^2 + \mu^2) - \mu^2 = \sigma^2 \quad \text{with probability 1 as } n \to \infty.
\]
4. Let $X_1, \ldots, X_{100}$ be independent Uniform$(0,e)$ random variables. Use the central limit theorem to find $c$ so that

$$P\left(\prod_{i=1}^{100} X_i \leq c\right) \approx \frac{1}{2}.$$ 

Solution: (5 marks) We first write

$$P\left(\prod_{i=1}^{100} X_i \leq c\right) = P\left(\sum_{i=1}^{100} \ln X_i \leq \ln c\right), = P\left(\sum_{i=1}^{100} Y_i \leq \ln c\right),$$

where we let $Y_i = \ln X_i$. To apply the central limit theorem we need to compute $E[Y_i]$. We will do this by first computing the pdf of $Y_i$. Let $X$ have a Uniform$(0,e)$ distribution and let $Y = \ln X$. The inverse transformation is $X = e^Y$. Then by the change of variable formula the pdf of $Y$ is

$$f_Y(y) = \begin{cases} \frac{1}{e} e^y & \text{for } y \leq 1 \\ 0 & \text{for } y > 1 \end{cases}$$

and the mean of $Y$ is

$$E[Y] = \frac{1}{e} \int_{-\infty}^{1} ye^y dy = \frac{1}{e} \left[ ye^y \big|_{-\infty}^{1} - \int_{-\infty}^{1} e^y dy \right] \quad \text{(integration by parts)}$$

$$= \frac{1}{e} \left[ e - e^1 \big|_{-\infty}^{-\infty} \right] = 0.$$ 

Since the mean of each $Y_i$ is 0, we just need $\ln c$ to equal 0, or $c = 1$. 