Queen’s University
Department of Mathematics and Statistics

MTHE/STAT 353
Homework 8 Solutions, 2021

• For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:

   (1) Use your own paper.
   (2) Use a tablet, such as an ipad.
   (3) Use document creation software, such as Word or LaTeX.

• Write your name and student number on the first page of each solution, and number your solution.

• For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 25
1. (4 marks) For each of the following parts, either give a distribution such that the given function is the mgf of that distribution, or show why the given function cannot be the mgf of any distribution.

(a) (2 marks) \( M(t) = \frac{1}{4}(1 + e^t)^2 \).

(b) (2 marks) \( M(t) = 1 + t^2 \).

Solution:

(a) We have \( M(t) = \frac{1}{4}(1 + 2e^t + e^{2t}) \). By inspection we can see that if \( X \) is a discrete random variable on \( \{0, 1, 2\} \) with probabilities \( P(X = 0) = \frac{1}{4}, P(X = 1) = \frac{1}{2}, \) and \( P(X = 2) = \frac{1}{4} \), then \( X \) will have mgf \( M(t) \).

(b) The function \( M(t) = 1 + t^2 \) cannot be the mgf of any distribution. If it were the mgf of some random variable \( X \), then we would have \( E[X] = M'(0) = 0, E[X^2] = M''(0) = 2, \) and \( E[X^n] = M^{(n)}(0) = 0 \). These moments would imply that \( \text{Var}(X^2) = E[X^4] - E[X^2]^2 = 0 - 2^2 = -4 \), which is impossible for a variance. Therefore, \( M(t) \) cannot be the mgf of any distribution.
2. (6 marks) Let $X_1, X_2, \ldots$ be a sequence of random variables such that $X_n$ has a Binomial distribution with parameters $n$ and $p_n$. Assume that the sequence $\{p_n\}_{n=1}^\infty$ satisfies $\lim_{n \to \infty} np_n = \lambda$, where $\lambda > 0$. Let $X$ have a Poisson distribution with parameter $\lambda$. Let $M_{X_n}(t)$ be the mgf of $X_n$ and let $M_X(t)$ be the mgf of $X$. Compute $M_{X_n}(t)$ and $M_X(t)$ and show that $M_{X_n}(t) \to M_X(t)$ as $n \to \infty$ for every $t$. You may use the fact that if $\{x_n\}$ is a sequence satisfying $x_n \to x$ as $n \to \infty$ then $(1 + \frac{x_n}{n})^n \to e^x$ as $n \to \infty$.

Solution: If we write $X_n = Y_1 + \ldots + Y_n$, where $Y_1, \ldots, Y_n$ are i.i.d. Bernoulli($p_n$) random variables, then the mgf of $X_n$ is given by $M_{X_n}(t) = (M_{Y}(t))^n$, where $M_{Y}(t)$ is the common mgf of each of the $Y_i$’s. Since $Y_i$ has a Bernoulli($p_n$) distribution, we have $M_{Y}(t) = e^{(0)t}(1 - p_n) + e^{(1)t}p_n = 1 - p_n + p_ne^t$. Thus, the mgf of $X_n$ is given by
\[
M_{X_n}(t) = (p_ne^t + 1 - p_n)^n = (1 - p_n(1 - e^t))^n = \left(1 - \frac{np_n(1 - e^t)}{n}\right)^n. \tag{1}
\]

For the Poisson($\lambda$) distribution we can directly compute the mgf as
\[
M_X(t) = \sum_{k=0}^\infty e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda} \sum_{k=0}^\infty \frac{(\lambda e^t)^k}{k!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}. \tag{2}
\]

In (1), as $n \to \infty$ we have that $np_n(1 - e^t) \to \lambda(1 - e^t)$ for every $t$, and we obtain that
\[
\left(1 - \frac{np_n(1 - e^t)}{n}\right)^n \to e^{-\lambda(1-e^t)} = e^{\lambda(e^t-1)}.
\]
as $n \to \infty$. The function on the right hand side above is the mgf of the Poisson($\lambda$) distribution, as given in (2).
3. (3 marks) Let $\mu > 0$ be given. Let $k$ be a positive integer. Give an example of a distribution with mean $\mu$ and finite variance such that if $X$ is a random variable with that distribution then

$$P(|X - E[X]| \geq k\sqrt{\text{Var}(X)}) = \frac{1}{k^2}.$$ 

(i.e., Chebyshev’s inequality is achieved).

Solution: Let $k \geq 1$ be a given positive integer. Let $X$ have a discrete distribution with the following pmf:

$$f(x) = \begin{cases} 
\frac{1}{2k^2} & \text{if } x = -k\sigma \\
1 - \frac{1}{k^2} & \text{if } x = 0 \\
\frac{1}{2k^2} & \text{if } x = k\sigma,
\end{cases}$$

where $\sigma > 0$ is arbitrary. We first verify that the variance of $X$ is $\sigma^2$. Since the distribution of $X$ is symmetric about 0 the mean of $X$ is 0, and so

$$\text{Var}(X) = E[X^2] = k^2\sigma^2 \left(\frac{1}{2k^2} + \frac{1}{2k^2}\right) = \sigma^2.$$ 

It is easy to verify that

$$P(|X - \mu| \geq k\sigma) = P(X = -k\sigma) + P(X = k\sigma) = \frac{1}{2k^2} + \frac{1}{2k^2} = \frac{1}{k^2}.$$
4. (6 marks) Let $X$ be a random variable with mgf $M_X(t)$, and suppose that $M_X(t)$ exists for all $t$.

(a) (3 marks) Using Markov’s inequality, show that for any $t > 0$ and any $a \in \mathbb{R}$,

$$P(X \geq a) \leq e^{-at}M_X(t).$$

(b) (3 marks) For any fixed $a$, minimizing the right hand side of the inequality in part (a) over $t > 0$ gives what is called the Chernoff bound to $P(X \geq a)$. If $X \sim N(0, 1)$, find Chernoff’s bound to $P(X \geq a)$ for $a > 0$.

Solution:

(a) Let $t > 0$ and $a \in \mathbb{R}$ be given. Then

$$P(X \geq a) = P(tX \geq ta) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-at}M_X(t),$$

where the inequality above is from Markov’s inequality.

(b) Let $a > 0$ be given. If $X \sim N(0, 1)$, then $M_X(t) = e^{t^2/2}$, and Chernoff’s bound is

$$P(X \geq a) \leq \min_{t>0} e^{-at} e^{t^2/2} = \min_{t>0} e^{(t^2-2at)/2}, \quad (3)$$

which is minimized over $t > 0$ by minimizing $t^2 - 2at$ over $t > 0$ since the exponential function is strictly increasing. Differentiating $t^2 - 2at$ with respect to $t$ and setting this to 0 gives $2t - 2a = 0$ or $t = a$, which is positive. Thus, the minimum over $t > 0$ is obtained at $t = a$ and plugging this into the right hand side of (3) gives

$$P(X \geq a) \leq e^{(a^2-2a^2)/2} = e^{-a^2/2}.$$
5. (6 marks) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ denote the sample mean of $X_1, \ldots, X_n$ for $n \geq 1$. Suppose that $\frac{1}{n} \sum_{i=1}^{n} \mu_i \to \mu$ as $n \to \infty$, for some $\mu \in \mathbb{R}$. Finally, suppose that $\sigma_i^2 \leq M$ for all $i \geq 1$, for some finite, positive $M$. Show that for any $\epsilon > 0$, $P(|\bar{X}_n - \mu| \geq \epsilon) \to 0$ as $n \to \infty$.

Solution: Let $\epsilon > 0$ be given. First note that $E[\bar{X}_n] = \frac{1}{n} \sum_{i=1}^{n} \mu_i$ and $\text{Var}(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2$. For ease of notation, let $\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \mu_i$. Using the triangle inequality we have

$$P(|\bar{X}_n - \mu| \geq \epsilon) = P(|\bar{X}_n - \bar{\mu}_n + \bar{\mu}_n - \mu| \geq \epsilon) \leq P(|\bar{X}_n - \bar{\mu}_n| + |\bar{\mu}_n - \mu| \geq \epsilon)$$

Since $\bar{\mu}_n \to \mu$ we can choose $N$ such that $|\bar{\mu}_n - \mu| < \epsilon$ for all $n \geq N$. Then for such $n$, since $E[\bar{X}_n] = \bar{\mu}_n$, and using Chebyshev’s inequality, we have

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq P(|\bar{X}_n - \bar{\mu}_n| \geq \epsilon - |\bar{\mu}_n - \mu|) \leq \frac{\text{Var}(\bar{X}_n)}{(\epsilon - |\bar{\mu}_n - \mu|)^2} = \frac{1}{n^2(\epsilon - |\bar{\mu}_n - \mu|)^2} \sum_{i=1}^{n} \sigma_i^2.$$ 

If there is an $M$ such that $\sigma_i^2 \leq M$ for all $i \geq 1$, then

$$\sum_{i=1}^{n} \sigma_i^2 \leq \sum_{i=1}^{n} M = Mn$$

and so

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{Mn}{n^2(\epsilon - |\bar{\mu}_n - \mu|)^2} = \frac{M}{n(\epsilon - |\bar{\mu}_n - \mu|)^2} \to 0 \quad \text{as } n \to \infty.$$ 

Thus, $\bar{X}_n$ converges to $\mu$ in probability as $n \to \infty$. 