Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Homework 9   Due April 11, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  (1) Use your own paper.
  (2) Use a tablet, such as an ipad.
  (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

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1. (5 marks) Show that if $X_n \to c$ in distribution, where c is a constant, then $X_n \to c$ in probability.

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2. (6 marks) Let $\Omega = (0,1)$ and P the Uniform(0,1) distribution on Ω. Define the random variables U_1 and U_2 by $U_1(\omega) = \omega$ and $U_2(\omega) = 1-\omega$, for $\omega \in \Omega$. Also, define the random variable Y by

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \geq \frac{1}{2} \\ 0 & \text{if } \omega < \frac{1}{2} \end{cases} .$$

Finally, let $X = YU_1 + (1-Y)U_2$ and $X_n = X$ for $n \geq 1$. Show that $X_n \to 1 - \frac{U_1}{2}$ in distribution but that X_n does not converge to $1 - \frac{U_1}{2}$ in probability.

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3. (6 marks) Let X and X_n, $n \geq 1$, be zero mean random variables. Let $\mathrm{Var}(X) = \sigma^2$ and $\mathrm{Var}(X_n) = \sigma_n^2$, and suppose that $\sigma_n^2 = \sigma^2$ for all n with $\sigma^2 < \infty$. Let $\rho_n = \rho(X_n, X)$ denote the correlation coefficient between X_n and X.

(a) (3 marks) If $\rho_n \to 1$ as $n \to \infty$, show that $X_n \to X$ in probability and in mean square.

(b) (3 marks) If $\rho_n \geq 1 - \frac{c}{n^2}$ for some positive constant c, show that $X_n \to X$ almost surely.

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4. (5 marks) Let $X_1, X_2, \dots$ be i.i.d. random variables with finite mean μ and finite variance σ^2. Show that the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ converges almost surely to σ^2, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is the sample mean of $X_1, \dots, X_n$.

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5. (6 marks) Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint: Apply the central limit theorem to the sequence $X_1, X_2, \ldots$, where the X_i are i.i.d. Poisson(1) random variables.