Queen's University
Department of Mathematics and Statistics

MTHE/STAT 353

Homework 9 Solutions, 2022

- For each question, your solution should start on a fresh page. You can write your solution using one of the following three formats:
  1. (1) Use your own paper.
  2. (2) Use a tablet, such as an ipad.
  3. (3) Use document creation software, such as Word or LaTeX.
- Write your name and student number on the first page of each solution, and number your solution.
- For each question, photograph or scan each page of your solution (unless your solution has been typed up and is already in electronic format), and combine the separate pages into a single file. Then upload each file (one for each question), into the appropriate box in Crowdmark.

Instructions for submitting your solutions to Crowdmark are also here.

Total Marks : 28

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1. (5 marks) Show that if $X_n \to c$ in distribution, where c is a constant, then $X_n \to c$ in probability.

Solution: Let F_n be the distribution function of X_n and let $F(x) = 0$ for $x < c$ and $F(x) = 1$ for $x \geq c$ (i.e., F is the distribution function of a random variable X that is equal to the constant c with probability 1). We are given that $X_n \to c$ in distribution, which is to say that $F_n(x) \to F(x)$ for every $x \neq c$. That is, $F_n(x) \to 0$ for every $x < c$ and $F_n(x) \to 1$ for every $x > c$. Now let $\epsilon > 0$ be given. To show that $X_n \to c$ in probability we need to show that $P(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$. But

$$\begin{aligned}
P(|X_n - c| > \epsilon) &= P(X_n < c - \epsilon) + P(X_n > c + \epsilon) \\
&\leq P(X_n \leq c - \epsilon) + P(X_n > c + \epsilon) \\
&= F_n(c - \epsilon) + 1 - F_n(c + \epsilon) \\
&\to 0 + 1 - 1 \quad \text{as } n \to \infty \\
&= 0.
\end{aligned}$$

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2. (6 marks) Let $\Omega = (0,1)$ and P the Uniform(0,1) distribution on Ω. Define the random variables U_1 and U_2 by $U_1(\omega) = \omega$ and $U_2(\omega) = 1-\omega$, for $\omega \in \Omega$. Also, define the random variable Y by

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \geq \frac{1}{2} \\ 0 & \text{if } \omega < \frac{1}{2} \end{cases}.$$

Finally, let $X = YU_1 + (1-Y)U_2$ and $X_n = X$ for $n \geq 1$. Show that $X_n \to 1 - \frac{U_1}{2}$ in distribution but that X_n does not converge to $1 - \frac{U_1}{2}$ in probability.

Solution: First consider the distribution of X. If $\omega \in [.5, 1)$ then $X(\omega) = U_1(\omega) = \omega$, while if $\omega \in (0, .5)$ then $X(\omega) = U_2(\omega) = 1-\omega$. Thus, for all $\omega \in \Omega$ we have $X(\omega) \in [.5, 1)$, which is the support of the distribution of X. For $x \in [.5, 1)$ we have

$$\begin{aligned} P(X \leq x) &= P(\{\omega \in \Omega : X(\omega) \leq x\}) \\ &= P(\{\omega \in [.5,1) : \omega \leq x\} \cup \{\omega \in (0,.5) : 1-\omega \leq x\}) \\ &= P([.5, x]) + P([1-x, .5)) = x - .5 + .5 - (1-x) = 2x - 1. \end{aligned}$$

For $x < .5$ we have $P(X \leq x) = 0$. For $x \geq 1$ we have $P(X \leq x) = 1$. So, the df of X is

$$F_X(x) = \begin{cases} 2x-1 & \text{if } .5 \leq x < 1 \\ 0 & \text{if } x < .5 \\ 1 & \text{if } x \geq 1 \end{cases}.$$

The pdf of X is $f_X(x) = 2$ if $x \in [.5, 1)$ and $f_X(x) = 0$ otherwise. So we see that X has a Uniform distribution on the interval $[.5, 1)$. On the other hand, it is clear that U_1 has a Uniform(0,1) distribution. Therefore, $P(1 - U_1/2 \leq x) = P(U_1 \geq 2(1-x)) = 1 - 2(1-x) = 2x - 1$ for $x \in [.5, 1)$. If $x < .5$ this probability is 0 and if $x \geq 1$ this probability is 1. So X and $1 - U_1/2$ have the same distribution, and since $X_n = X$ for all n we have trivially that X_n converges to $1 - U_1/2$ in distribution. Now we show that X_n does not converge to $1 - U_1/2$ in probability. Let $\epsilon > 0$ be given. Then

$$\begin{aligned} P(|X_n - (1 - U_1/2)| \geq \epsilon) &= P(|X - (1 - U_1/2)| \geq \epsilon) \\ &= P(\{\omega \in \Omega : |X(\omega) - (1 - U_1(\omega)/2)| \geq \epsilon\}) \\ &\geq P(\{\omega \in [.5,1) : |X(\omega) - (1 - U_1(\omega)/2)| \geq \epsilon\}) \\ &= P(\{\omega \in [.5,1) : |\omega - (1-\omega)/2)| \geq \epsilon\}) \\ &= P(\{\omega \in [.5,1) : |1.5\omega - .5| \geq \epsilon\}). \end{aligned}$$

But if $\epsilon \leq .25$ then $|1.5\omega - .5| \geq \epsilon$ for all $\omega \in [.5, 1)$. So, $P(|X_n - (1 - U_1/2)| \geq \epsilon) \geq 1/2$ for all n, for $\epsilon \in (0, .25]$. Hence, X_n does not converge to $1 - U_1/2$ in probability.

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3. (6 marks) Let X and X_n, $n \geq 1$, be zero mean random variables. Let $\text{Var}(X) = \sigma^2$ and $\text{Var}(X_n) = \sigma_n^2$, and suppose that $\sigma_n^2 = \sigma^2$ for all n with $\sigma^2 < \infty$. Let $\rho_n = \rho(X_n, X)$ denote the correlation coefficient between X_n and X.

(a) (3 marks) If $\rho_n \to 1$ as $n \to \infty$, show that $X_n \to X$ in probability and in mean square.

(b) (3 marks) If $\rho_n \geq 1 - \frac{c}{n^2}$ for some positive constant c, show that $X_n \to X$ almost surely.

Solution:

(a) To show that $X_n \to X$ in probability we can use Chebyshev's inequality. Let $\epsilon > 0$ be given. Since $E[X_n - X] = 0$ for all n, we have

$$\begin{aligned} P(|X_n - X| \geq \epsilon) &\leq \frac{\text{Var}(X_n - X)}{\epsilon^2} \quad \text{(by Chebyshev's inequality)} \\ &= \frac{\sigma_n^2 + \sigma^2 - 2\sigma_n \sigma \rho_n}{\epsilon^2} \\ &= \frac{2\sigma^2(1-\rho_n)}{\epsilon^2} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

if $\rho_n \to 1$ as $n \to \infty$. Therefore, $X_n \to X$ in probability. For convergence in mean square, note that $E[|X_n - X|^2] = \text{Var}(X_n - X)$, and the same inequalities as above show that $E[|X_n - X|^2] \to 0$ as $n \to \infty$. Therefore, $X_n \to X$ in mean square.

(b) If $\rho_n \geq 1 - \frac{c}{n^2}$ then $1 - \rho_n \leq \frac{c}{n^2}$. Then

$$P(|X_n - X| \geq \epsilon) \leq \frac{2\sigma^2(1-\rho_n)}{\epsilon^2} \leq \frac{2Mc}{\epsilon^2 n^2},$$

where the first inequality follows as in part(a). The last inequality above implies that $\sum_{n=1}^{\infty} P(|X_n - X| \geq \epsilon) < \infty$, and this holds for any $\epsilon > 0$. But then $X_n \to X$ almost surely by the sufficient condition from the lectures.

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4. (5 marks) Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ and finite variance σ^2. Show that the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ converges almost surely to σ^2, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is the sample mean of $X_1, \ldots, X_n$.

Solution: Write S^2 as

$$\begin{aligned} S^2 &= \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2 \\ &= \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - 2\overline{X}_n \sum_{i=1}^n X_i + n\overline{X}_n^2\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\overline{X}_n^2\right) \\ &= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) - \left(\frac{n}{n-1}\right)\overline{X}_n^2. \end{aligned}$$

By the strong law of large numbers $\frac{1}{n}\sum_{i=1}^n X_i^2 \to E[X_1^2] = \sigma^2 + \mu^2$ almost surely as $n \to \infty$. Also by the strong law of large numbers $\overline{X}_n \to \mu$ almost surely as $n \to \infty$. Since $f(\overline{X}_n) = \overline{X}_n^2$ is a continuous function of $\overline{X}_n$, we have that $\overline{X}_n^2 \to \mu^2$ almost surely as $n \to \infty$. Letting

$$\begin{aligned} A_1 &= \left\{\omega : \frac{1}{n}\sum_{i=1}^n X_i^2(\omega) \to \sigma^2 + \mu^2 \quad \text{as } n \to \infty\right\} \\ A_2 &= \{\omega : \overline{X}_n^2(\omega) \to \mu^2 \quad \text{as } n \to \infty\} \end{aligned}$$

and $B = A_1 \cap A_2$, we have that $P(B) = 1$. Since $\frac{n}{n-1} \to 1$ as $n \to \infty$, for $\omega \in B$ we have

$$\left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_{i=1}^n X_i^2(\omega)\right) - \left(\frac{n}{n-1}\right)\overline{X}_n^2(\omega) \to (1)(\sigma^2 + \mu^2) - (1)(\mu^2) = \sigma^2$$

as $n \to \infty$. Since $P(B) = 1$ we have that S^2 converges to σ^2 almost surely.

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5. (6 marks) Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint: Apply the central limit theorem to the sequence $X_1, X_2, \ldots$, where the X_i are i.i.d. Poisson(1) random variables.

Solution: Let $S_n = X_1 + \ldots + X_n$. Each X_i has a Poisson(1) distribution, so each X_i has mean 1 and variance 1. On the one hand, by the central limit theorem, $(S_n - n)/\sqrt{n}$ has an approximate $N(0,1)$ distribution. On the other hand, from the lectures we know that the exact distribution of S_n is Poisson(n) (or see Theorem 11.5, p.469). From the exact distribution,

$$P(S_n \le n) = \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n},$$

while from the approximate distribution,

$$P(S_n \le n) = P\left(\frac{S_n - n}{\sqrt{n}} \le \frac{n-n}{\sqrt{n}}\right) \approx P(Z \le 0) = \frac{1}{2},$$

where $Z \sim N(0,1)$. In the limit as $n \to \infty$ the approximation above is exact. Equating the two, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n} = \frac{1}{2}.$$