Multiple Integrals Review

Say \( f(x_1, \ldots, x_n) \) is a real-valued function of \( n \) variables and \( A \subset \mathbb{R}^n \), and we wish to compute the multiple integral
\[
\int_A f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n
\]
The limits of integration on the inner integrals can depend on the values of the variables for the outer integrals.

\[\text{e.g.} \begin{array}{|c|c|}
\hline
x_1 & \overbrace{\hphantom{\int}} A \\
\hline
\end{array}\]

If you do the integrals over \( x_1 \) and \( x_2 \) in the order
\[
\int_a^b \int_c^d f(x_1, x_2) \, dx_1 \, dx_2
\]
the limits of integration will depend on the values of \( x_2 \).

In general,

1. Pick an order in which to do the iterated integrals wisely (i.e., try to make finding limits of integration and/or doing the iterated integrals simpler).
2. Figure out the limits of integration. Suppose you have chosen the order of doing the integrals as
   - integrate over the variable \( x_{i_1} \) as the inner integral
   - \( x_{i_2} \) as the next innermost integral
   - \( x_{i_n} \) as the outer integral
   i.e., you will perform the \( n \)-dimensional integral as
   \[
   \int_A \ldots \int_A f(x_1, \ldots, x_n) \, dx_{i_1} \, dx_{i_2} \ldots dx_{i_n}
   \]
   - vary \( (x_1, \ldots, x_n)^\top \) over \( A \), then all possible values of \( x_{i_n} \) as you do this will give the limits of the outer integral.
   - next, find the limits of the second outermost integral over \( x_{i_n-1} \). To do this for each fixed value of \( x_{i_n} \), vary \( (x_1, \ldots, x_n)^\top \) over \( A \) with \( x_{i_n} \) fixed and find all the possible values of \( x_{i_n-1} \), which in general will be a function of \( x_{i_n} \). This gives the limits for the second outermost integral.
In general, when finding the limits of integration for variable $X_i$, you have found the limits of integration for the variables $X_{in}, \ldots, X_{ij+1}$. To do this for fixed values of $X_{in}, \ldots, X_{ij+1}$, vary $(X_1, \ldots, X_n)$ over $A$ with $X_{in}, \ldots, X_{ij+1}$ fixed and find all the possible values of $X_i$, which in general will be a function of $X_{in}, \ldots, X_{ij+1}$. These possible values of $X_i$ give the limits of integration for the variable $X_i$.

**Example** Suppose that $(X_1, X_2, X_3)^T$ are jointly continuous and have joint pdf
\[
f_X(X_1, X_2, X_3) = \begin{cases} 1 & \text{for } 0 \leq X_1, X_2, X_3 \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]
What is the probability that the random quadratic equation $X_1 y^2 + X_2 y + X_3$ has real roots?

**Sol'n**. The roots are $\frac{-X_2 \pm \sqrt{X_2^2 - 4X_1X_3}}{2X_1}$. The roots are real if and only if $X_2^2 - 4X_1X_3 \geq 0$. So we want
\[
P((X_1, X_2, X_3)^T \in A), \text{ where } A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2^2 - 4x_1x_3 \geq 0\}
\]
Say we choose the ordering $dX_2 dX_1 dX_3$. The integral can be written as
\[
\int \int \int_{A} I_{[0,1]^3}(x_1, x_2, x_3) \, dX_2 \, dX_1 \, dX_3,
\]
where
\[
I_{[0,1]^3}(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } 0 \leq x_1, x_2, x_3 \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]
Constraints:
\[
\begin{align*}
x_2^2 - 4x_1x_3 \geq 0, \\
0 \leq x_1 \leq 1, \\
0 \leq x_2 \leq 1, \\
0 \leq x_3 \leq 1
\end{align*}
\]

(*
As I vary \((x_1, x_2, x_3)\) over all possible values satisfying the constraints \((*)\), the possible values of \(x_1\) are in \([0, 1]\).

So the limits of the outer integral are from 0 to 1.

Next, we find the limits for \(x_1\) for each given value of \(x_3 \in [0, 1]\).

If \(x_3 \in [0, \frac{1}{4}]\) then \(4x_1x_3\) will be in \([0, 1]\) for all \(x_1 \in [0, 1]\).

If \(x_3 \in [\frac{1}{4}, 1]\) then \(x_1\) must be \(\leq \frac{1}{4}x_3\) for \(x_1^2 - 4x_1x_3 \geq 0\) to be possible. So the triple integral should be broken up into 2 cases:

\[
\int_0^1 \int_0^{\frac{1}{4}x_3} \int_0^1 \, dx_2 \, dx_1 \, dx_3 + \int_0^1 \int_0^{\frac{1}{4}x_3} \int_{\frac{1}{4}x_3}^1 \, dx_2 \, dx_1 \, dx_3
\]

Lastly, for given values of \(x_1\) and \(x_3\) in their limits of integration, the possible values of \(x_2\) are from \(\sqrt{4x_1x_3}\) to 1. So we get the integral

\[
\int_0^1 \int_0^{\frac{1}{4}x_3} \left( \int_0^1 (1 - 4x_1x_3) \, dx_3 \right) \, dx_1
\]

\[
= \int_0^1 \left( \frac{x_3^{3/2}}{3/2} \right) \, dx_3
\]

\[
= \left[ x_3^{3/2} \right]_0^{1/4} + \left[ \frac{1}{4} \ln x_3 - \frac{3}{8} \ln x_3 \right]_0^{1/4}
\]

\[
= (\frac{1}{4} - \frac{8}{9} + \frac{4}{3} + \frac{9}{36}) - (\frac{1}{4} - \frac{8}{9} + \frac{9}{36})
\]

\[
= \frac{1}{4} - \frac{1}{4} + \ln 4 \left( \frac{3}{12} - \frac{2}{12} \right)
\]

\[
= \frac{9 - 4}{36} + \frac{1n 4}{12} = \frac{5}{36} + \frac{3 \ln 4}{36} = \frac{1}{36} (5 + 6 \ln 2)
\]

\[
\approx 0.2544
\]