Review for Final Exam, Apr. 8

Part 1
Questions 1-4 of 2016 Final Exam.

Review for Final Exam, Apr. 9

Part 2
Questions 5-6 of 2016 Exam.
Questions 3, 5, 6 of 2013 Exam.

Question involving m.g.f.s and the strong law of large numbers! Let $X_1, X_2, \ldots$ be i.i.d. random variables, each with mean $\mu$ and common m.g.f. $M_X(t)$.

Show that $M_X(\frac{t}{n})^n \to e^\mu$ as $n \to \infty$.

Solution:

\[
M_X(\frac{t}{n})^n = M_{X_1}(\frac{t}{n}) \times \cdots \times M_{X_n}(\frac{t}{n})
\]

where $M_{X_i}(t)$ is the m.g.f. of $X_i$.

\[
= M_{X_1 + \cdots + X_n}(\frac{t}{n})
\]

\[
= E\left[e^{\frac{t}{n}(X_1 + \cdots + X_n)}\right]
\]

\[
= E\left[e^{\bar{X}_n}\right]
\]

but $\bar{X}_n \to \mu$ a.s. by the strong law of large numbers, and $e^{\bar{X}_n} \to e^\mu$ as $n \to \infty$ since $e^x$ is a continuous function.

\[
\to E(e^{\mu}) = e^\mu
\]
Queen’s University
Department of Mathematics and Statistics

MTHE/STAT 353
Final Examination    April 16, 2016
Instructor: G. Takahara

• “Proctors are unable to respond to queries about the interpretation of exam ques-
tions. Do your best to answer exam questions as written.”

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any assumptions made if doubt exists as to the interpretation of any question that
requires a written answer.”

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constitute a breach of academic integrity under the University Senates Academic
Integrity Policy Statement.

• Formulas and tables are attached.

• An 8.5 × 11 inch sheet of notes (both sides) is permitted. Simple calculators are
permitted (Casio 991, red, blue, or gold sticker). HOWEVER, do reasonable sim-
plifications.

• Write the answers in the space provided, continue on the backs of pages if needed.

• SHOW YOUR WORK CLEARLY. Correct answers without clear work showing
how you got there will not receive full marks.

• Marks per part question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]       Problem 4 [10]
Problem 2 [10]       Problem 5 [10]
Problem 3 [10]       Problem 6 [10]

Total: [60]
1. Let $X, Y, Z$ be jointly continuous random variables with joint pdf

$$f(x, y, z) = \begin{cases} \frac{cxyz}{c} & \text{for } 0 < x < y < z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

for some normalizing constant $c$.

(a) Find $c$.

$$1 = \int_0^1 \int_0^x \int_0^z cxyz \, dz \, dy \, dx$$

$$= c \int_0^1 \int_0^x \frac{y^2}{2} \, dy \, dx$$

$$= c \left[ \frac{1}{3} x^2 y^2 \right]_0^1 = c \frac{1}{3} = \frac{c}{48}$$

$$\Rightarrow c = 48$$

(b) Compute $P(Y \leq 1/2)$.

$$P(Y \leq \frac{1}{2})$$

$$= 48 \int_0^{1/2} \int_0^y \frac{y^2}{2} \, dx \, dy$$

$$= 48 \int_0^{1/2} \frac{y^3}{2} \left[ \int_0^y dx \right] \, dy$$

$$= 48 \int_0^{1/2} \frac{y^3}{2} \left( \frac{1}{2} - \frac{y^2}{2} \right) \, dy$$

$$= 48 \int_0^{1/2} \frac{y^3}{4} \left( 1 - y^2 \right) \, dy$$

$$= 48 \left[ \frac{1}{4} \left( \frac{1}{16} \right)^{1/2} - \frac{1}{24 \times 64} \right]$$

$$= 48 \left( \frac{1}{16} - \frac{1}{24 \times 64} \right) = \frac{3}{16} - \frac{1}{32} = \frac{5}{32}.$$
(b) (extra worksheet if needed)
2. Let \((X_1, Y_1, Z_1)^T, \ldots, (X_{10}, Y_{10}, Z_{10})^T\) be 10 random vectors in \(\mathbb{R}^3\) that are mutually independent and discrete, each with joint pmf

\[ p(x, y, z) = \begin{cases} \frac{x+y+z}{12} & \text{for } x \in \{0, 1\}, y \in \{0, 1\}, z \in \{0, 1\} \\ 0 & \text{otherwise.} \end{cases} \]

Let \(Z_i\) be the number these vectors whose components sum to \(i\), for \(i = 1, 2, 3\). Find \(P(Z_1 = 2, Z_2 = 5, Z_3 = 3)\). [10]

Here, we have 10 multinomial trials. In the \(i\)th trial, \((X_i, Y_i, Z_i)\) is generated and the sum of the components is recorded. Possible outcomes of each trial are 0, 1, 2, or 3.

\[
\begin{array}{c|c|c}
(X_i, Y_i, Z_i) & P(X_i, Y_i, Z_i) & P(\text{outcome of trial is } i) \\
\hline
0 0 0 & 0 & 0 \\
0 0 1 & 1/12 & 1 \quad 3/12 = 1/4 \\
0 1 0 & 1/12 & 2 \quad 6/12 = 1/2 \\
0 1 1 & 2/12 & 3 \quad 3/12 = 1/4 \\
1 0 0 & 1/12 & \\
1 0 1 & 2/12 & \\
1 1 0 & 2/12 & \\
1 1 1 & 3/12 & \\
\end{array}
\]

\((Z_1, Z_2, Z_3)\) has a multinomial distribution with parameters \(n = 10\), \(p_1 = \frac{1}{4}\), \(p_2 = \frac{1}{2}\), \(p_3 = \frac{1}{4}\).

\[
P(Z_1 = 2, Z_2 = 5, Z_3 = 3)
= \binom{10}{2,5,3} \left(\frac{1}{4}\right)^2 \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)^3
= \frac{10!}{2!5!3!} \cdot \frac{1}{2^{15}} = \frac{\frac{6 \times 7 \times 8 \times 9 \times 10}{2 \times 6}}{2^{15}} = \frac{1}{2^{15}}
= \frac{4 \times 7 \times 9 \times 10}{2^{15}} = \frac{630}{2^{13}} = \frac{315}{2^{12}}
\]
3. Let $X_1, \ldots, X_n$ be independent random variables, each with an Exponential distribution with mean 1. Let $X_{(1)}$ and $X_{(2)}$ denote the first and second order statistics, respectively. Find $\text{Cov}(e^{X_{(1)}}, e^{X_{(2)}})$.

Each $X_i$ has pdf $f(x) = e^{-x}$, $x > 0$.

and cdf $F(x) = 1 - e^{-x}$, $x > 0$

pdf of $X_{(1)}$ is

$$f_1(x_1) = n (1 - F(x_1))^{n-1} f(x_1) = n (e^{-x_1})^{n-1} e^{-x_1} = n e^{-nx_1}, \quad x_1 > 0$$

pdf of $X_{(2)}$ is

$$f_2(x_2) = F(x_2) f(x_2) (1 - F(x_2))^{n-2} n (n-1) = (1 - e^{-x_2}) e^{-x_2} (e^{-x_2})^{n-2} n (n-1) = n (n-1) (1 - e^{-x_2}) e^{-(n-1)x_2}, \quad x_2 > 0$$

Joint pdf of $(X_{(1)}, X_{(2)})$ is

$$f_{12}(x_1, x_2) = n (n-1) f(x_1) f(x_2) (1 - F(x_2))^{n-2} = n (n-1) e^{-x_1} e^{-x_2} (e^{-x_2})^{n-2} = n (n-1) e^{-x_1} e^{-(n-1)x_2}, \quad 0 < x_1 < x_2$$

Then

$$\text{Cov}(e^{X_{(1)}}, e^{X_{(2)}}) = E[e^{X_{(1)} + X_{(2)}}] - E[e^{X_{(1)}}]E[e^{X_{(2)}}]$$

$$E[e^{X_{(1)} + X_{(2)}}] = \int_0^\infty \int_0^\infty e^{x_1 + x_2} n (n-1) e^{-x_1} e^{-(n-1)x_2} \, dx_2 \, dx_1$$

$$= n (n-1) \int_0^\infty \left[ \int_0^{x_2} e^{-(n-2)x_2} \, dx_2 \right] \, dx_1$$
3. (extra worksheet, if needed)

\[
= n(n-1) \int_0^\infty \left( e^{-\frac{(n-1)x_2}{2}} \right) dx_1
\]

\[
= \frac{n(n-1)}{n-2} \int_0^\infty e^{-(n-2)x_1} dx_1
\]

\[
= \frac{n(n-1)}{(n-2)^2} \int_0^\infty (n-2) e^{-(n-2)x_1} dx_1 = \frac{n(n-1)}{(n-2)^2}
\]

\[
E[e^{x_1}] = \int_0^\infty e^{x_1} n e^{-x_1} dx_1 = n \int_0^\infty e^{-(n-1)x_1} dx_1
\]

\[
= \frac{n}{n-1} \int_0^\infty (n-1) e^{-(n-1)x_1} dx_1
\]

\[
E[e^{x_2}] = \int_0^\infty e^{x_2} n(n-1)(1-e^{-x_2}) e^{-(n-1)x_2} dx_2
\]

\[
= n(n-1) \left[ \int_0^\infty e^{-(n-2)x_2} dx_2 - \int_0^\infty e^{-(n-1)x_2} dx_2 \right]
\]

\[
= n(n-1) \left[ \frac{1}{n-2} - \frac{1}{n-1} \right]
\]

\[
= \frac{n(n-1)}{(n-1)(n-2)} = \frac{n}{n-2}
\]

So, finally,

\[
\text{Cov}(e^{x_1}, e^{x_2}) = \frac{n(n-1)}{n-2} - \frac{(n(n-1))(n-2)}{n-2}
\]

\[
= \frac{n(n-1)(n-1) - n^2(n-2)}{(n-1)(n-2)^2}
\]

\[
= \frac{n(n^2-2n+1) - n^3 + 2n^2}{n(n-1)(n-2)^2}
\]

\[
= \frac{n}{(n-1)(n-2)^2}
\]
4(a). A triangle has vertices labelled 0, 1 and 2. A particle moves randomly among the vertices as follows. From its current location it will move to the adjacent vertex clockwise with probability \( p \) and to the adjacent vertex counterclockwise with probability \( 1 - p \), where \( p \in (0, 1) \) is fixed. If the particle starts at vertex 0 compute the expected number of moves until the particle returns to vertex 0. 

**Hint:** Let \( T_0 \) be the desired expected number of moves and let \( T_i , i = 1, 2, \) denote the expected number of moves to first reach vertex 0 starting from vertex \( i \). By conditioning set up a system of equations for \( T_0 , T_1 , \) and \( T_2 \).

\[
\begin{align*}
\text{Condition on first move of the particle.} & \\
\text{Get } T_0 &= p(1 + T_1) + (1-p)(1 + T_2) \quad (1) \\
T_1 &= p(1 + T_2) + (1-p)(1) \quad (2) \\
T_2 &= p(1) + (1-p)(1 + T_1) \quad (3)
\end{align*}
\]

\( (3) \Rightarrow T_2 = 1 + (1-p)T_1 \)

Plug into \( (2) \):

\[
T_1 = 1 + p(1 + (1-p)T_1)
\]

\[
\Rightarrow T_1 = \frac{1 + p}{1-p(1-p)}
\]

Then

\[
T_2 = 1 + (1-p) \frac{1 + p}{1-p(1-p)} = \frac{1 - p(1-p) + (1-p)(1+p)}{1-p(1-p)}
\]

\[
= \frac{1 - p + p^2 + 1 - p^2}{1-p(1-p)}
\]

\[
= \frac{2-p}{1-p(1-p)}
\]

Then from \( (1) \), get

\[
T_0 = p\left(1 + \frac{1 + p}{1-p(1-p)}\right) + (1-p)\left(1 + \frac{2-p}{1-p(1-p)}\right)
\]

\[
= \left(1 + \frac{p(1+p) + (1-p)(2-p)}{1-p(1-p)}\right)
\]

\[
= \frac{1 + p(1+p) + (1-p)(2-p)}{1-p(1-p)}
\]
(b) Let \( p \in (0,1) \) be fixed. A coin with probability of heads \( p \) is flipped until a heads is obtained. Let \( Y \) denote the number of flips required. The coin is then flipped \( Y \) times. Let \( X \) denote the number of heads obtained in these \( Y \) flips. Find the mean and variance of \( X \).

\[ Y \sim \text{Geometric}(p) \]
\[ X \mid Y = y \sim \text{Binomial}(y, p) \]

\[
E[X] = E(E[X \mid Y]) = E[\rho Y] = p E[Y] = p \left( \frac{1}{p} \right) = 1
\]

\[
\text{Var}(X) = E[\text{Var}(X \mid Y)] + \text{Var}(E[X \mid Y])
\]
\[
= E[\gamma \rho (1-\rho)] + \text{Var}(\rho Y)
\]
\[
= \rho (1-\rho) E[Y] + \rho^2 \text{Var}(Y)
\]
\[
= 1-\rho + \rho^2 \frac{1-\rho}{\rho^2} = 2(1-\rho)
\]
5(a). Let $Y_n$ have an Exponential($n$) distribution for $n \geq 1$. Let $X$ be an arbitrary random variable and let $X_n = X + Y_n$. Show that $X_n$ converges to $X$ almost surely. [5]

Let $\varepsilon > 0$ be given.

$$P(|X_n - X| > \varepsilon) = P(|X + Y_n - X| > \varepsilon)$$
$$= P(|Y_n| > \varepsilon)$$
$$= P(Y_n > \varepsilon)$$
$$= e^{-ne^\varepsilon}$$

So

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) = \sum_{n=1}^{\infty} e^{-ne^\varepsilon}$$
$$= \sum_{n=1}^{\infty} (e^{-e^\varepsilon})^n \quad \text{(note } e^{-e^\varepsilon} \in (0,1))$$
$$= \frac{1}{1 - e^{-e^\varepsilon}} - 1 < \infty$$

So $X_n \to X$ a.s. by the sufficient condition from class.
(b) Let \( \{X_n\}_{n=1}^{\infty} \) and \( \{Y_n\}_{n=1}^{\infty} \) be two sequences of random variables and suppose that \( X_n \) converges to \( X \) in probability and \( Y_n \) converges to \( Y \) in probability, for some random variables \( X \) and \( Y \). Show that \( X_n + Y_n \) converges to \( X + Y \) in probability. \textit{Hint:} Use the triangle inequality and the fact that if \( a \) and \( b \) are positive numbers then \( a + b > \epsilon \) implies at least one of \( a > \epsilon/2 \) or \( b > \epsilon/2 \) must hold.

Let \( \epsilon > 0 \) be given.

\[
P( |X_n + Y_n - (X + Y)| > \epsilon )
= P( |X_n - X + Y_n - Y| > \epsilon )
\leq P( |X_n - X| + |Y_n - Y| > \epsilon ) \quad \text{(by triangle inequality)}
\leq P( |X_n - X| > \frac{\epsilon}{2} ) + P( |Y_n - Y| > \frac{\epsilon}{2} )
\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{since } X_n \xrightarrow{p} X \text{ and } Y_n \xrightarrow{p} Y.

Therefore, \( X_n + Y_n \xrightarrow{p} X + Y \).
6(a). Let $X_1, X_2, \ldots$ be a sequence of random variables, each having a Uniform(0,1) distribution, and let $X$ be a random variable, also with a Uniform(0,1) distribution, independent of all the $X_i$'s. Show that $X_n$ does not converge almost surely to $X$ as $n \to \infty$.

We show that $X_n$ does not converge to $X$ in probability. Let $\varepsilon > 0$ be given, $P(\{X_n - X > \varepsilon\}) = (1 - \varepsilon)^2 \to 0$ as $n \to \infty$. Thus $X_n \not\to X$. This implies that $X_n \not\to X$.

(b) Let $(\Omega, P)$ be a probability space. Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ be a decreasing sequence of sets in $\Omega$ and let $A = \bigcap_{n=1}^{\infty} A_n$. Let

$$X_n(\omega) = I_{A_n}(\omega) = \begin{cases} 1 & \text{if } \omega \in A_n \\ 0 & \text{if } \omega \notin A_n \end{cases} \quad \text{and} \quad X(\omega) = I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Show that $X_n$ converges to $X$ almost surely.

If $\omega \in A$ then $\omega \in \bigcap_{n=1}^{\infty} A_n$, i.e., $\omega \in A_n$ for every $n$. Then $X_n(\omega) = 1$ for every $n$. Therefore, $X_n(\omega) \to 1 = X(\omega)$ as $n \to \infty$. If $\omega \notin A$ then $\omega \notin A_m$ for some $m$. But since $\{A_n\}$ is a decreasing sequence we have that $\omega \notin A_n$ for all $n \geq m$. So $X_n(\omega) = 0$ for all $n \geq m$. Therefore, $X_n(\omega) \to 0 = X(\omega)$ as $n \to \infty$. So $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$.

So $X_n \overset{a.s.}{\to} X$. 

So $X_n \to X$.
Formula Sheet

Special Distributions

Beta distribution with parameters $\alpha > 0$ and $\beta > 0$:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \frac{\alpha}{\alpha + \beta}, \quad \text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

$\alpha = 1$ and $\beta = 1$ gives the Uniform distribution on $(0, 1)$.

Gamma distribution with parameters $r > 0$ and $\lambda > 0$:

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1}e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \frac{r}{\lambda}, \quad \text{Var}(X) = \frac{r}{\lambda^2}$$

$$M_X(t) = \left( \frac{\lambda}{\lambda - t} \right)^r \quad \text{for } t < \lambda.$$

$r = 1$ gives the Exponential distribution with mean $1/\lambda$.

Geometric distribution with parameter $p \in (0, 1)$:

$$f(k) = \begin{cases} p(1-p)^{k-1} & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}$$

$$M_X(t) = \frac{pe^t}{1 - (1-p)e^t} \quad \text{for } t \neq -\ln(1-p).$$

Binomial distribution with parameters $n$ and $p \in (0, 1)$:

$$f(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = np, \quad \text{Var}(X) = np(1-p)$$

$$M_X(t) = (p + (1-p)e^t)^n \quad \text{for } t \in \mathbb{R}.$$
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Problem 2 [10] Problem 5 [10]

Problem 3 [10] Problem 6 [10]

Total: [60]
1. Suppose we have $B$ empty boxes initially. An experiment consists of choosing a box at random and then placing a ball inside the box. Suppose this experiment is repeated independently $2n$ times, first using $n$ red balls and then using $n$ blue balls (so the $B$ boxes now contains $2n$ balls in total). Write an expression (which may be left as a sum) for the expected number of boxes that contain the same number of red and blue balls. [10]
2. (a) If $X$ and $Y$ are two continuous random variables with correlation coefficient $\rho$, explain why $(X, Y)$ cannot have a joint density if $|\rho| = 1$. [3]

(b) A fair die is rolled twice. The rolls are independent. Let

$$X = \begin{cases} 
1 & \text{if the sum of the two rolls is greater than 6} \\
0 & \text{otherwise}
\end{cases}$$

and let $Y$ be the number of sixes rolled. Compute the correlation coefficient between $X$ and $Y$. [7]
3. Let $X$ have a Gamma($\lambda, 1$) distribution and $Y$ have a Poisson($\lambda$) distribution ($\lambda > 0$), and $X$ and $Y$ are independent.

(a) For $k > \lambda$, apply Chebyshev's inequality to show that $P(X > k) \leq \lambda/(k - \lambda)^2$. [3]

$$P(X > k) = P(X - \lambda > k - \lambda) \quad \text{(note } k - \lambda > 0)$$

$$\leq P(|X - \lambda| > k - \lambda)$$

$$\leq \frac{\text{Var}(X)}{(k - \lambda)^2} \quad \text{(by Chebyshev)}$$

$$= \frac{\lambda}{(k - \lambda)^2}$$

(b) Using conditioning and applying Markov's inequality, show that $P(X > Y + 1) \leq 1 - e^{-\lambda}$. [7]

$$P(X > Y + 1) = \sum_{n=0}^{\infty} P(X > Y + 1 \mid Y = n) P(Y = n)$$

$$= \sum_{n=0}^{\infty} P(X > n + 1) \frac{\lambda^n}{n!} e^{-\lambda}$$

$$\leq \sum_{n=0}^{\infty} \frac{E[X]}{n+1} \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{(by Markov)}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda}{n+1} \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{(n+1)!}$$

$$= e^{-\lambda} (e^\lambda - 1)$$

$$= 1 - e^{-\lambda}$$
4. Suppose an urn contains 6 balls, $k$ of which are red and $6 - k$ are blue. At each draw, a ball is drawn at random and replaced by a ball of the other colour (if a red ball is drawn it is replaced by a blue ball and vice versa). For $k \neq 3$, let $r_k$ denote the expected number of draws until the urn contains 3 red and 3 blue balls. For $k = 3$, let $r_3$ denote the expected number of draws until the urn returns to having 3 red and 3 blue balls. Compute $r_3$. [10]
5. Let $X_1, X_2, \ldots$ be a sequence of random variables such that $X_n \in [0, 1]$ for all $n$ and $X_n$ has cdf $F_n(x) = x^n$ for $x \in [0, 1]$ (and $F_n(x) = 0$ for $x < 0$ and $F_n(x) = 1$ for $x > 1$). Show that $X_n \to 1$ in the $r$th mean, for any positive integer $r$, and that $X_n \to 1$ with probability 1.

\[ E[|X_n - 1|^r] = E[(1 - X_n)^r] \quad \text{since } X_n \leq 1 \text{ w.p. 1.} \]
\[ = \tau_1 \int_0^1 (1 - x)^r x^{n-1} \, dx \quad \text{(the pdf of } X_n \text{ is } x^{n-1} \text{ for } x \in [0, 1]) \]
\[ = n \int_0^1 x^{n-1} (1-x)^{(r+1)-1} \, dx \]
\[ = \frac{n \Gamma(n) \Gamma(r+1)}{\Gamma(n+r+1)} \int_0^1 \frac{\Gamma(n+r+1)}{\Gamma(n) \Gamma(r+1)} x^{n-1} (1-x)^{(r+1)-1} \, dx \]
\[ = \frac{n(n-1)! \cdot r!}{(n+r)!} = \frac{r!}{(n+1)x \ldots x(n+r)} \to 0 \text{ as } n \to \infty \]

\[ \sum_{n=1}^{\infty} P(|X_n - 1| \geq \varepsilon) = \sum_{n=1}^{\infty} (1 - \varepsilon)^n = \frac{1}{1 - (1 - \varepsilon)} - 1 \]
\[ = \frac{1 - \varepsilon}{\varepsilon} < \infty \]

By sufficient condition for convergence a.s. we have $X_n \Rightarrow 1$. 

Let $\varepsilon > 0$ be given.

\[ P(1 |X_n - 1| > \varepsilon) = P(1 - X_n > \varepsilon) = P(X_n < 1 - \varepsilon) \]
\[ = F_n(1 - \varepsilon) \]
\[ = (1 - \varepsilon)^n \]

\[ \sum_{n=1}^{\infty} P(1 |X_n - 1| > \varepsilon) = \sum_{n=1}^{\infty} (1 - \varepsilon)^n = \frac{1}{1 - (1 - \varepsilon)} - 1 \]
\[ = \frac{1 - \varepsilon}{\varepsilon} < \infty \]

By sufficient condition for convergence a.s. we have $X_n \Rightarrow 1$. 

\[\]
6. Let $X_1, \ldots, X_n$ by independent random variables, each with a Uniform\((-k, k)\) distribution (so each $X_i$ has pdf $f(x) = \frac{1}{2k} I_{[-k,k]}(x)$). Use the central limit theorem to approximate the probability

$$P\left(|X_1^r + \ldots + X_n^r| > kp\right),$$

where $r$ is an odd positive integer and $p$ is a positive real number. Express your answer in terms of $k$, $p$, $r$, $n$ and $\Phi$, the cumulative distribution function of the standard normal distribution. Based on this approximation, for a given $r$, for what values of $p$ does this probability go to 0 as $k \to \infty$?

$$\frac{X_1^r, \ldots, X_n^r}{\sqrt{n/2k}} \overset{d}{\to} N(0,1)$$

Then

$$P\left(|X_1^r + \ldots + X_n^r| > kp\right) = P\left(\frac{\sqrt{k(2r+1)}}{\sqrt{n}} |X_1^r + \ldots + X_n^r| > \frac{\sqrt{k(2r+1)}}{\sqrt{n}} kp\right) \Rightarrow 1 \geq \Phi \left( \frac{\sqrt{k(2r+1)} kp + \frac{k}{2r+1}}{\sqrt{n}} \right)$$

Since $\Phi \left( \frac{\sqrt{k(2r+1)} kp + \frac{k}{2r+1}}{\sqrt{n}} \right) \to 0$ as $k \to \infty$ for all $p > 0$ and so

$$\Phi \left( \frac{\sqrt{k(2r+1)} (kp + \frac{1}{2r+1})}{\sqrt{n}} \right) \to 1$$

as $k \to \infty$ and $(*) \to 0$. \[10\]
Formula Sheet

Special Distributions

Gamma with parameters $\alpha$ and $\beta$:
\[
f(x) = \begin{cases} 
\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} & \text{if } x > 0 \\
0 & \text{otherwise.}
\end{cases}
\]
\[E[X] = \frac{\alpha}{\beta}, \quad \text{Var}(X) = \frac{\alpha}{\beta^2}.
\]

Beta with parameters $\alpha$ and $\beta$:
\[
f(x) = \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{-\beta-1} & \text{if } 0 < x < 1 \\
0 & \text{otherwise.}
\end{cases}
\]
\[E[X] = \frac{\alpha}{\alpha+\beta}, \quad \text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.
\]

Poisson with parameter $\lambda$:
\[
f(k) = \begin{cases} 
\frac{\lambda^k e^{-\lambda}}{k!} & \text{if } k = 0, 1, \ldots \\
0 & \text{otherwise.}
\end{cases}
\]
\[E[X] = \lambda, \quad \text{Var}(X) = \lambda.
\]

Normal (Gaussian) with mean 0 and variance 1:
\[
f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.
\]