If $A$ is an event then we can express $P(A)$ as an expectation by defining the indicator $I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$.

$E[I_A] = P(A)$. Thus, we can apply the law of total expectation.

If $Y$ is a random vector then we can write

$$P(A) = E[I_A] = E[E[I_A | Y)]$$

$$= \left\{ \begin{array}{ll}
\sum_y E[I_A | Y=y] P(Y=y) & \text{(discrete case)} \\
\int_y E[I_A | Y=y] f_Y(y) & \text{(continuous case)}
\end{array} \right.$$

$$= \left\{ \begin{array}{ll}
\sum_y P(A | Y=y) P(Y=y) & \text{(discrete case)} \\
\int_y P(A | Y=y) f_Y(y) & \text{(continuous case)}
\end{array} \right.$$  

**Example** Say we have a stick of length 1. The stick is broken at 2 randomly selected points on the stick. What is the probability that the resulting 3 pieces of the stick can be used to form a triangle.

**Solution** Let $X$ and $Y$ denote the 2 randomly selected points. Then $X$ and $Y$ are independent Uniform(0,1) random variables. Let $A$ denote the event that the resulting 3 pieces can be used to form a triangle. We first note that a triangle can be formed from the 3 pieces if and only if the length of the longest piece is less than 1/2. Let us compute $P(A)$ by conditioning on $X$. Suppose $X=x$ is given, where $x \in (0,1)$.

If $x \in (0,.5]$.

From picture, the length of the longest piece will be $< .5$ if and only if $y \in (1/2, x+.5)$.

If $x \in (0.5, 1)$ then the picture is

From picture, the length of the longest piece will be $< .5$ if and only if $y \in (x-.5, .5)$.

And the length of the longest piece will be $< .5$ if and only if $y \in (x-.5, .5)$. 
Then, by the law of total expectation
\[
P(A) = \int_0^1 P(A \mid X = x) \, dx
\]
\[
= \int_0^{.5} P(A \mid X = x) \, dx + \int_{.5}^1 P(A \mid X = x) \, dx
\]
\[
= \int_0^{.5} P(Y \in (x, x+.5)) \, dx + \int_{.5}^1 P(Y \in (x-.5, x)) \, dx
\]
\[
= \int_0^{.5} .5 \, dx + \int_{.5}^1 (1-x) \, dx
\]
\[
= \left[ \frac{x^2}{2} \right]_0^{.5} - \left[ \frac{(1-x)^2}{2} \right]_{.5}^1 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\]

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Taking stock of where we are: There are roughly 3 weeks left in the course. My plan is to start now with Chapter 11 (moment generating functions, sums of independent random variables, Markov and Chebyshev inequalities, Laws of large numbers, Central limit theorem). I want to get through this material in roughly 2 weeks and the last 2 homeworks will be on this material. In the last 2 or so lectures of the course I will go back to Chapter 10 and discuss the Multivariate Normal distribution. There will be no homework problems on this and it will not be covered in the final exam (final exam is April 24, 2pm-5pm).
Moment Generating Function

**Definition** Let $X$ be a random variable. The moment generating function of $X$, denoted by $M_X(t)$, is defined as:

$$M_X(t) = E[e^{tX}]$$

for all $t \in \mathbb{R}$ for which the expectation exists.

**Example** Say $X \sim \mathcal{N}(0,1)$. Then

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2 - t^2} dx$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

$$= e^{t^2/2}$$

This expectation exists for all $t \in \mathbb{R}$.

Therefore, the mgf of the $\mathcal{N}(0,1)$ distribution is $M_X(t) = e^{t^2/2}$, $t \in \mathbb{R}$. 
