Since the mgf of a random variable $X$ is $M_X(t) = E[e^{tx}]$, it is determined by the distribution of $X$. So if $X$ and $Y$ are 2 random variables with the same distribution, then $M_X(t) = M_Y(t)$. Under certain conditions, the converse is true, which we state in the following theorem (without proof).

**Theorem** Suppose that $M_X(t)$ and $M_Y(t)$ are mgfs such that $M_X(t) = M_Y(t)$ for all $t$ in an open interval $(-\delta_0, \delta_0)$ for some $\delta_0 > 0$. Then $F_X(z) = F_Y(z)$ for all $z \in \mathbb{R}$, where $F_X$ and $F_Y$ are the df's corresponding to the distributions associated with $M_X(t)$ and $M_Y(t)$.

Then we can conclude that if the mgf $M_X(t)$ of a random variable $X$ exists in some open interval about 0, then it is unique. Thus, in this case the mgf is another way to represent the distribution of a random variable.

**Properties of mgfs**

1. $M_X(0) = E[e^{0X}] = E[e^0] = E[1] = 1$.
2. $\frac{d^n}{dt^n} M_X(t) = \frac{d^n}{dt^n} E[e^{tx}]$
   
   $= E\left[\frac{d^n}{dt^n} e^{tx}\right]$
   
   $= E[X^n e^{tx}]$

   Then $\frac{d^n}{dt^n} M_X(t) \bigg|_{t=0} = E[X^n]$.

So, the moments of the distribution of $X$ can be obtained in this way.

3. Say $X_1, \ldots, X_n$ are mutually independent and suppose $X_i$ has mgf $M_{X_i}(t)$, $i = 1, \ldots, n$. Let $X = X_1 + \ldots + X_n$. Then the mgf of $X$ is

   $M_X(t) = E[e^{tx}] = E[e^{tx(X_1+\ldots+X_n)}]$

   $= E\left[\prod_{i=1}^n e^{tx_i}\right]$

   $= \prod_{i=1}^n E[e^{tx_i}] = \prod_{i=1}^n M_{X_i}(t)$.
Let \( X \) be a random variable and let \( Y = aX+b \) where \( a \) and \( b \) are constants. Then the mgf of \( Y \) is
\[
M_Y(t) = E[e^{tY}] = E[e^{t(ax+b)}]
= e^{tb} E[e^{tX}]
= e^{tb} M_X(at)
\]

Example We saw last time that if \( X \sim N(0,1) \), then
\[
M_X(t) = e^{t^2/2}
\]
If \( Y = \sigma X + \mu \) for some \( \sigma > 0 \) and \( \mu \in \mathbb{R} \), then \( Y \sim N(\mu, \sigma^2) \). By property \( 4 \), we get that the mgf of \( Y \) is
\[
M_Y(t) = e^{t\mu} M_X(\sigma t) = e^{t\mu} e^{\sigma^2 t^2/2} = e^{t\mu + \frac{1}{2} \sigma^2 t^2},
\]
which exists for all \( t \in \mathbb{R} \).

Example Let \( X \sim \text{Gamma}(r, \lambda) \). Then
\[
M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{\lambda^r}{r!} x^{r-1} e^{-\lambda x} \, dx
= \frac{\lambda^r}{r!} \int_0^\infty x^{r-1} e^{-x(\lambda-t)} \, dx
\]
This integral exists for all \( t < \lambda \), so we can write it as
\[
= \frac{\lambda^r}{r! (\lambda-t)^r} \int_0^\infty \frac{(\lambda-t)^r}{r!} x^{r-1} e^{-x(\lambda-t)} \, dx
= \left( \frac{\lambda}{\lambda-t} \right)^r,
\]
for \( t < \lambda \).

Now suppose that \( X_1, \ldots, X_n \) are independent with \( X_i \sim \text{Gamma}(r_i, \lambda) \) and let \( X = X_1 + \ldots + X_n \). By property \( 3 \), the mgf of \( X \) is
\[
M_X(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \left( \frac{\lambda}{\lambda-t} \right)^{r_i} = \left( \frac{\lambda}{\lambda-t} \right)^{\sum_{i=1}^n r_i}
\]
This is the mgf of a \( \text{Gamma}(r_1+\ldots+r_n, \lambda) \). By uniqueness of the mgf, the distribution of \( X_1 + \ldots + X_n \) must be \( \text{Gamma}(r_1+\ldots+r_n, \lambda) \).
Example Suppose that \( X \sim \text{Poisson}(\lambda) \). The mgf of \( X \) is
\[
M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} \frac{e^{tx}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)},
\]
which exists for all \( t \in \mathbb{R} \).

Now suppose \( X_1, \ldots, X_n \) are independent with \( X_i \sim \text{Poisson}(\lambda_i) \).

Let \( X = X_1 + \cdots + X_n \). By property 2, the mgf of \( X \) is
\[
M_X(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t-1)} = e^{(\sum_{i=1}^{n} \lambda_i)(e^t-1)}.
\]
This is the mgf of a \( \text{Poisson}(\lambda_1 + \cdots + \lambda_n) \). By the uniqueness of mgf's, the distribution of \( X_1 + \cdots + X_n \) must \( \text{Poisson}(\lambda_1 + \cdots + \lambda_n) \).