Strong Law of Large Numbers

Recall, the weak law of large numbers says that if \( X_1, X_2, \ldots \) are i.i.d. with finite mean \( m \) and finite variance \( \sigma^2 \), and 
\[
\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i ,
\]
then \( \bar{X}_n \xrightarrow{p} m \).

Theorem (Strong Law of Large Numbers).

Let \( X_1, X_2, \ldots \) be i.i.d. random variables with finite mean \( m \) and finite variance \( \sigma^2 \). Let 
\[
\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i .
\]
Then \( \overline{X}_n \overset{a.s.}{\to} m \).

Proof The proof is in 3 steps.

① Consider the subsequence of \( \{ \overline{X}_n \} \) given by \( \{ \overline{X}_{n^2} \} \), i.e.,
\[
\overline{X}_{n^2}, \overline{X}_{2n^2}, \overline{X}_{3n^2}, \ldots ,
\]
Let \( \varepsilon > 0 \) be given.

Then 
\[
P \left( \left| \overline{X}_{n^2} - m \right| \geq \varepsilon \right) \leq \frac{\text{Var}(\overline{X}_{n^2})}{\varepsilon^2}
\]
(by Chebyshev's inequality)
\[
= \frac{\text{Var} \left( \frac{1}{n^2} \sum_{i=1}^{n^2} X_i \right)}{\varepsilon^2}
\]
\[
= \frac{\frac{1}{n^4} n^2 \sigma^2}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2 n^2}
\]
Then 
\[
\sum_{n=1}^{\infty} P \left( \left| \overline{X}_{n^2} - m \right| \geq \varepsilon \right) \leq \frac{\sigma^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
\]

By the sufficient condition for convergence a.s., we have that 
\[
\overline{X}_{n^2} \overset{a.s.}{\to} m .
\]

② Now assume that \( X_i \geq 0 \) (i.e., \( X_i(\omega) \geq 0 \) for all \( \omega \in \Omega \)).
Let 
\[
S_n = X_1 + \ldots + X_n .
\]
Since the \( X_i \) are nonnegative, the sequence \( \{ S_n \} \) is nondecreasing.
Furthermore, for any \( n \) there is a unique \( i \) such that \( i^2 \leq n < (i+1)^2 \) (as \( n \) increases, the \( i \) defined in this way also increases with \( n \)).
Then, since \( \{ S_n^2 \} \) is nondecreasing,
\[
S_{i+2}^2 \leq S_n \leq S_{(i+1)^2}
\]
Since \( i^2 \leq n \leq (i+1)^2 \), we can write
\[
\frac{S_i^2}{(i+1)^2} \leq \frac{S_n}{n} \leq \frac{S_{(i+1)^2}}{i^2}
\]
\[
\implies \frac{i^2}{(i+1)^2} \frac{S_i^2}{i^2} \leq \frac{S_n}{n} \leq \frac{S_{(i+1)^2}}{(i+1)^2} \frac{(i+1)^2}{i^2}
\]
\[
\implies \frac{(i+1)^2}{i^2} \frac{S_i^2}{i^2} \leq \frac{S_n}{n} \leq \frac{S_{(i+1)^2}}{(i+1)^2} \frac{(i+1)^2}{i^2}
\]
As \( n \to \infty \), \( \frac{(i+1)^2}{i^2} \to 1 \) and \( \frac{(i+1)^2}{i^2} \to 1 \) . Now, let
\[
A = \{ \omega \in \Omega : \overline{X}_n^2(\omega) \to M \text{ as } n \to \infty \}
\]
In step 1 we showed that \( P(A) = 1 \). If \( \omega \in A \),
\[
\frac{(i+1)^2}{i^2} \overline{X}_i^2(\omega) \leq \overline{X}_n(\omega) \leq \overline{X}_{(i+1)^2}(\omega) \frac{(i+1)^2}{i^2}
\]
for all \( n \)
\[
\implies \liminf_{n \to \infty} \overline{X}_n(\omega) \leq \limsup_{n \to \infty} \overline{X}_n(\omega) \leq M
\]
Letting \( n \to \infty \) we get
\[
M \leq \liminf_{n \to \infty} \overline{X}_n(\omega) \leq \limsup_{n \to \infty} \overline{X}_n(\omega) \leq M
\]
\[
\implies \liminf_{n \to \infty} \overline{X}_n(\omega) = \limsup_{n \to \infty} \overline{X}_n(\omega)
\]
\[
\implies \lim_{n \to \infty} \overline{X}_n(\omega) \text{ exists and is equal to } M.
\]
Since this holds for any \( \omega \in A \) and \( P(A) = 1 \), we have \( \overline{X}_n \to_f M \).

3) Now we remove the nonnegativity assumption on the \( X_i \)'s.
Write \( X_i \) as \( X_i = X_i^+ - X_i^- \), where
\[
X_i^+ = \max(X_i, 0) \quad \text{and} \quad X_i^- = \max(-X_i, 0)
\]
For every \( \omega \) we get \( X_i(\omega) = X_i^+(\omega) - X_i^-(\omega) \). Also, the \( X_i^+ \)
and \( X_i^- \) are all nonnegative random variables.
Since the $X_i$'s are i.i.d., then so are $X_i^+ = \max(X_i, 0)$ and $X_i^- = \max(-X_i, 0)$.

Let $\mu^+ = E[X_i^+]$ and $\mu^- = E[X_i^-]$.

Then

\[\mu = E[X_i] = E[X_i^+ - X_i^-] = E[X_i^+] - E[X_i^-] = \mu^+ - \mu^-\]

Let $A^+ = \{\omega \in \Omega: \frac{1}{n}\sum_{i=1}^{n} X_i^+(\omega) \to \mu^+ \text{ as } n \to \infty\}$

$A^- = \{\omega \in \Omega: \frac{1}{n}\sum_{i=1}^{n} X_i^-(\omega) \to \mu^- \text{ as } n \to \infty\}$.

By step 2, $P(A^+) = 1$ and $P(A^-) = 1$. Let $B = A^+ \cap A^-$. Then $P(B) = 1$. For $\omega \in B$,

\[\frac{1}{n}\sum_{i=1}^{n} X_i^+(\omega) - \frac{1}{n}\sum_{i=1}^{n} X_i^-(\omega) = \frac{1}{n}\sum_{i=1}^{n} X_i^+(\omega) - \frac{1}{n}\sum_{i=1}^{n} X_i^+(\omega) - \frac{1}{n}\sum_{i=1}^{n} X_i^-(\omega) + \frac{1}{n}\sum_{i=1}^{n} X_i^+(\omega) - \frac{1}{n}\sum_{i=1}^{n} X_i^-(\omega) \to \mu^+ - \mu^- = \mu \text{ as } n \to \infty\]

So $X_n(\omega) \to \mu$ for $\omega \in B$. Since $P(B) = 1$, we conclude that $X_n \xrightarrow{a.s.} \mu$. 