Marginal cdf's:

Let $X_1, \ldots, X_n$ have joint cdf $F_X(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$.

Let $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and

$$\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$$

($i_1, \ldots, i_k$ are distinct).

The joint marginal cdf of $(X_{i_1}, \ldots, X_{i_k})$ is

$$F_{X_{i_1}, \ldots, X_{i_k}}(x_{i_1}, \ldots, x_{i_k}) = P(X_{i_1} \leq x_{i_1}, \ldots, X_{i_k} \leq x_{i_k})$$

$$= P(X_{i_1} \leq x_{i_1}, \ldots, X_{i_k} \leq x_{i_k}, X_{j_1} < \infty, \ldots, X_{j_{n-k}} < \infty)$$

$$= \lim_{x_{j_1} \to \infty} \lim_{x_{j_{n-k}} \to \infty} P(X_{i_1} \leq x_{i_1}, \ldots, X_{i_k} \leq x_{i_k}, X_{j_1} \leq x_{j_1}, \ldots, X_{j_{n-k}} \leq x_{j_{n-k}})$$

i.e. Take full joint cdf and let arguments for $X_{j_1}, \ldots, X_{j_{n-k}}$ go to $\infty$. 

Independence of Multiple Random Variables

First, let us look at $n$ events $A_1, \ldots, A_n$. What do we mean for them to be mutually independent?

Intuitively (and mathematically), we want $P(A_1 \mid A_2) = P(A_1)$, where $P(A_1 \mid A_2)$ is the conditional probability of $A_1$ given $A_2$.

Then $P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)}$

So, if $A_1$ and $A_2$ are independent, then we have $\frac{P(A_1 \cap A_2)}{P(A_2)} = P(A_1)$

or $P(A_1 \cap A_2) = P(A_1)P(A_2)$
Now consider $n$ events $A_1, ..., A_n$.

**Definition** $A_1, ..., A_n$ are mutually independent if

$$P(A_1 \cap \ldots \cap A_n) = P(A_1) \times \ldots \times P(A_n)$$

and

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \times \ldots \times P(A_{i_k})$$

for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

**Remark.** Pairwise independence does not imply mutual independence.

**Ex 1**

$$P(\text{region}) = \text{area of region}$$

$$= \text{area of intersection of 2 circles}$$

But intersection of 3 circles is empty and \((\text{area of circle})^3 > 0\) so

$$0 = P(A_1 \cap A_2 \cap A_3) \neq P(A_1)P(A_2)P(A_3) > 0$$

**Ex 2**

Every pair has same intersection.
\[ P(\{ X_2 = X_3 \}) \]
\[ = P(\bigcup_{\kappa=1}^{\infty} \{ X_2 = X_3 = \kappa \}) \]
\[ = \sum P( X_2 = X_3 = \kappa) \]
\[ = \sum P( X_2 = \kappa, X_3 = \kappa) \]
\[ = \sum P( X_1 = 0^3) \text{ or } \{ X_2 = 0 \} \text{ or } \{ X_2 = X_3 \} \]
Definition: We say that $n$ random variables $X_1, \ldots, X_n$ are mutually independent if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \times \ldots \times P(X_n \in A_n)$$

for any $A_1, \ldots, A_n \subset \mathbb{R}$.

Note that we can take any of the $A_i$ to be $\mathbb{R}$ so if we take $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and $\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$, then

$$P(X_{i_1} \in A_{i_1}, \ldots, X_{i_k} \in A_{i_k})$$

$$= P(X_{i_1} \in A_{i_1}, \ldots, X_{i_k} \in A_{i_k}, X_{j_1} \in \mathbb{R}, \ldots, X_{j_{n-k}} \in \mathbb{R})$$

$$= P(X_{i_1} \in A_{i_1}) \times \ldots \times P(X_{i_k} \in A_{i_k}) \times (1) \times \ldots \times (1)$$

So any subset of $X_1, \ldots, X_n$ are mutually independent if $X_1, \ldots, X_n$ are mutually independent.
Remarks

1. If $X_1, \ldots, X_n$ are mutually independent then so are $g_1(X_1), \ldots, g_n(X_n)$ where $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$.

Proof

$$P(g_1(X_1) \in A_1, \ldots, g_n(X_n) \in A_n)$$

$$= P(X_1 \in g_1^{-1}(A_1), \ldots, X_n \in g_n^{-1}(A_n))$$

where $g_i^{-1}(A_i) = \{x \in \mathbb{R} : g_i(x) \in A_i\}$.

$$= P(X_1 \in g_1^{-1}(A_1)) \times \ldots \times P(X_n \in g_n^{-1}(A_n))$$

$$= P(g_1(X_1) \in A_1) \times \ldots \times P(g_n(X_n) \in A_n)$$

2. The $X_i$'s in the definition of independence can be any random quantities (e.g., random vectors or random matrices). The only change is that the $A_i$'s must be in the appropriate range space of $X_i$. 

Theorem. \( X_1, \ldots, X_n \) are mutually independent if and only if
\[
f_X(x_1, \ldots, x_n) = f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n)
\]
(continuous case) \[\leadsto\]
joint pdf \[\mapsto\]
marginal pdfs

or
\[
p_X(x_1, \ldots, x_n) = p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n)
\]
(discrete case) \[\mapsto\]
joint pmf \[\mapsto\]
marginal pmfs

Proof. Suppose the above factoring holds.

Let \( A_1, \ldots, A_n \subset \mathbb{R} \) be arbitrary. Then
\[
P(X_1 \in A_1, \ldots, X_n \in A_n)
= \int \cdots \int f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \quad \text{(cont's case)}
\]
\[
\left\{ \begin{array}{l}
\sum_{x_n \in A_n} \sum_{x_1 \in A_1} p_X(x_1, \ldots, x_n) \quad \text{(discrete case)}
\end{array} \right.
\]
\[
= \int \cdots \int f_{X_1}(x_1) \cdots f_{X_n}(x_n) \, dx_1 \cdots dx_n
\]
\[
\left\{ \begin{array}{l}
\sum_{x_n \in A_n} \sum_{x_1 \in A_1} p_{X_1}(x_1) \cdots p_{X_n}(x_n)
\end{array} \right.
\]
\[
\begin{align*}
&= \left( \int_{A_n} f_{x_n}(x_n) \, dx_n \right) \times \ldots \times \left( \int_{A_1} f_{x_1}(x_1) \, dx_1 \right) \\
&= \left( \sum_{x_n \in A_n} p_{x_n}(x_n) \right) \times \ldots \times \left( \sum_{x_1 \in A_1} p_{x_1}(x_1) \right)
\end{align*}
\]

Now assume that \( X_1, \ldots, X_n \) are mutually independent:

**Discrete case**

By independence

\[
P(X_1 = x_1, \ldots, X_n = x_n) = p(x_1) \times \ldots \times p(x_n)
\]

**Continuous case**

By independence

\[
P(X_1 \leq x_1, \ldots, X_n \leq x_n) = p(x_1) \times \ldots \times p(x_n)
\]

where \( F_x \) is the joint cdf and \( F_{x_i} \) are marginal cdfs.

But the

\[
f_x(x_1, \ldots, x_n) = \frac{d^n}{dx_1 \cdots dx_n} F_x(x_1, \ldots, x_n)
\]

\[
= \frac{d^n}{dx_1 \cdots dx_n} F_{x_1}(x_1) \times \ldots \times F_{x_n}(x_n)
\]
Theorem: \( X_1, \ldots, X_n \) are mutually independent if and only if
\[
F_x(x_1, \ldots, x_n) = F_{X_1}(x_1) \times \ldots \times F_{X_n}(x_n)
\]
where \( F_x \) is joint cdf and \( F_{X_i} \) is marginal cdf of \( X_i, i = 1, \ldots, n \).

Proof: Suppose \( X_1, \ldots, X_n \) are mutually independent. Then
\[
F_x(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(X_1 \leq x_1) \times \ldots \times P(X_n \leq x_n) = F_{X_1}(x_1) \times \ldots \times F_{X_n}(x_n)
\]

Now suppose
\[
F_x(x_1, \ldots, x_n) = F_{X_1}(x_1) \times \ldots \times F_{X_n}(x_n)
\]
for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).

In the case when \( X_1, \ldots, X_n \) are jointly continuous, then from proof of previous theorem we get that
\[
f_x(x_1, \ldots, x_n) = f_{X_1}(x_1) \times \ldots \times f_{X_n}(x_n),
\]
which then implies that \( X_1, \ldots, X_n \) are mutually independent. To prove that

\[
= f_{X_1}(x_1) \ldots f_{X_n}(x_n)
\]
\( F_x(x_1, \ldots, x_n) = F_{x_1}(x_1) \times \cdots \times F_{x_n}(x_n) \) implies that \( X_1, \ldots, X_n \) are mutually independent is outside the scope of this course.

**Expectation Involving Multiple Random Variables**

Expectation is only defined for random variables. If \( X = (X_1, \ldots, X_n)^T \) is a random vector then we may write \( E[X] \), which only mean

\[
\begin{bmatrix}
E[X_1] \\
\vdots \\
E[X_n]
\end{bmatrix}
\]

Now, each \( E[X_i] \) is with respect to the marginal distribution of \( X_i \). However, it is not necessary to compute each marginal distribution.

**Theorem** If \( g(x_1, \ldots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R} \) is a real-valued function of \( x_1, \ldots, x_n \), then

\[
E[g(X_1, \ldots, X_n)] = \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f_X(x_1, \ldots, x_n) dx_1 \ldots dx_n \\
\left\{ \begin{array}{c}
\sum_{x_1} \ldots \sum_{x_n} g(x_1, \ldots, x_n) p_X(x_1, \ldots, x_n) \\
\end{array} \right\}
\]
Proof (discrete case)

Let \( Y = g(X_1, \ldots, X_n) \). Then

\[
E[g(X_1, \ldots, X_n)] = E(Y)
= \sum_y y \cdot P(Y = y)
= \sum_y \sum_{x: g(x) = y} P(X = x)
= \sum_x g(x) \cdot P(X = x)

Example. Suppose \((X_1, X_2, X_3)\) have joint pdf

\[
f(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1 - x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2 - x_3)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_3^2}{2}}
\]

(cf. problem 2, hmk 1)

Compute \( E(X_1 X_2) \). In our heads, we get 1

integrate over \( X_1 \) first, then \( X_2 \), then \( X_3 \).