Marginal pdfs

Suppose \( X_1, \ldots, X_n \) are jointly continuous with joint pdf \( f_x(x_1, \ldots, x_n) \). Let \( \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\} \) with \( d < n \). We are interested in the joint distribution of the subset \( X_{i_1}, \ldots, X_{i_d} \).

Let \( \{j_1, \ldots, j_{n-d}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_d\} \). A joint pdf of \( X_{i_1}, \ldots, X_{i_d} \), say \( f_{i_1, \ldots, i_d}(x_{i_1}, \ldots, x_{i_d}) \), is any function from \( \mathbb{R}^d \) to \([0, \infty)\) satisfying

\[
\int_{\mathbb{R}^d} f_{i_1, \ldots, i_d}(x_{i_1}, \ldots, x_{i_d})\,dx_{i_1} \ldots dx_{i_d} = P((X_{i_1}, \ldots, X_{i_d})^T \in A), \text{ for any } A \subset \mathbb{R}^d.
\]

If we have the full pdf \( f_x(x_1, \ldots, x_n) \) of \( X_1, \ldots, X_n \), then for \( A \subset \mathbb{R}^d \) we can write

\[
P((X_{i_1}, \ldots, X_{i_d})^T \in A) = P((X_{i_1}, \ldots, X_{i_d})^T \in A, (X_{j_1}, \ldots, X_{j_{n-d}})^T \in \mathbb{R}^{n-d})
\]

\[
\int_{A} \int_{\mathbb{R}^{n-d}} f_x(x_1, \ldots, x_{i_d}, x_{j_1}, \ldots, x_{j_{n-d}})\,dx_{j_1} \ldots dx_{j_{n-d}}\,dx_{i_1} \ldots dx_{i_d}
\]

this is a function of \( X_{i_1}, \ldots, X_{i_d} \)

(since we've integrated out the variable \( X_{j_1}, \ldots, X_{j_{n-d}} \) and it is nonnegative.

Then we can conclude that a joint pdf for \( X_{i_1}, \ldots, X_{i_d} \) is given by

\[
f_{i_1, \ldots, i_d}(x_{i_1}, \ldots, x_{i_d}) = \int_{\mathbb{R}^{n-d}} f_x(x_1, \ldots, x_n)\,dx_{j_1} \ldots dx_{j_{n-d}}.
\]

Remark: For each \( X_{i_1}, \ldots, X_{i_d} \) one must be careful about which values of \( X_{j_1}, \ldots, X_{j_{n-d}} \) give a positive value for \( f_x(x_1, \ldots, x_n) \) because this will affect the limits of integration.

Example. Suppose \( (X, Y, Z)^T \in \mathbb{R}^3 \) are jointly continuous with joint pdf \( f(x, y, z) = \begin{cases} c & \text{for } 0 \leq z \leq 1, -1-z \leq x \leq 1-z, \text{ and } -1-z \leq y \leq 1-z \\ 0 & \text{otherwise} \end{cases} \)

where \( c \) is a positive constant.

First, let us find \( c \).
\( c \) must satisfy:
\[
1 = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} c \, dx \, dy \, dz = c \int_{-1}^{1} \left( 1 - z^2 \right)^2 \, dz = 4c \left( \frac{1}{3} \right)^3 = 4c \frac{1}{3} \Rightarrow c = \frac{3}{4}.
\]

Next, find the joint marginal pdf of \((X, Y)\). For given values of \(x \) and \(y \) in \([-1, 1]\), the constraints on \(z \) are:
\[
\begin{align*}
X - 1 \leq z &\Rightarrow z \leq 1 - X \\
-(1 - z) \leq X &\Rightarrow z \leq 1 + X \\
\text{Also, we have} & \\
z &\leq 1 - y \\
z &\leq 1 + y
\end{align*}
\]
Can combine these constraints as:
\[
2 \leq 1 - |x| \\
z \leq 1 - |y|
\]
which can be further written succinctly as:
\[
z \leq 1 - \max(|x|, |y|).
\]

Then, for \(x, y \in [-1, 1]\), we have the joint marginal pdf of \((X, Y)\) at \((x, y)\) is given by:
\[
f_{XY}(x, y) = \int_{1 - \max(|x|, |y|)}^{1} \frac{3}{4} \, dz = \frac{3}{4} \left( 1 - \max(|x|, |y|) \right).
\]
So:
\[
f_{XY}(x, y) = \begin{cases} 
\frac{3}{4} \left( 1 - \max(|x|, |y|) \right) & \text{for } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]