Order Statistics

Let \( X_1, \ldots, X_n \) be jointly continuous, mutually independent, and identically distributed. The \( k \)th order statistic, denoted by \( X_{(k)} \), is
\[ X_{(k)} = k^{\text{th}} \text{ smallest among } X_1, \ldots, X_n, \ k = 1, \ldots, n \]
The two most commonly used order statistics are
\[ X_{(1)} = \min(X_1, \ldots, X_n) = \min_i X_i \]
\[ X_{(n)} = \max(X_1, \ldots, X_n) = \max_i X_i \]

Let \( f(x) \) be the common marginal pdf of each \( X_i, \ i = 1, \ldots, n \)
Let \( F(x) \) be the common marginal df of each \( X_i, \ i = 1, \ldots, n \)
Let \( f_X \) and \( F_X \) denote the pdf and df, respectively, of the \( k \)th order statistic.

Distribution of \( X_{(n)} \)
The usual way the df and pdf of \( X_{(n)} \) is computed is
\[ F_n(x) = P(\max X_i \leq x) \]
\[ = P(X_1 \leq x, \ldots, X_n \leq x) \]
\[ = P(X_1 \leq x) \times \cdots \times P(X_n \leq x) \text{ by mutual independence} \]
\[ = F(x)^n \]
The pdf of \( X_{(n)} \) is then
\[ f_n(x) = \frac{d}{dx} F_n(x) = \frac{d}{dx} F(x)^n = n F(x)^{n-1} F'(x) = n F(x)^{n-1} f(x) \]

Example If the \( X_i \)'s are Uniform \((0,1)\), then
\[ F(x) = \begin{cases} \frac{x}{1} & 0 \leq x < 1 \\ 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \end{cases} \]
and
\[ f(x) = \begin{cases} \frac{1}{1} & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \]
The pdf of \( X_{(n)} \) is
\[ f_n(x) = \begin{cases} n x^{n-1} & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \]
Distribution of $X_{(1)}$

Similarly to how we computed the df of $X_{(n)}$, we have

\[
F_1(x) = P(\min X_i \leq x) = 1 - P(\min X_i > x)
\]

\[
= 1 - P(X_1 > x, \ldots, X_n > x)
\]

\[
= 1 - P(X_1 > x) \cdots P(X_n > x)
\]

\[
= 1 - (1 - F(x))^n
\]

Then the pdf of $X_{(1)}$ is

\[
f_1(x) = \frac{\frac{d}{dx} F_1(x)}{F_1(x)} = \frac{\frac{d}{dx} [1 - (1 - F(x))^n]}{1 - (1 - F(x))^n} = n(1 - F(x))^{n-1} \frac{d}{dx} (-F(x))
\]

\[
= n(1 - F(x))^{n-1} f(x)
\]

Example If $X_1, \ldots, X_n$ are i.i.d. Uniform $(0, 1)$, then $X_{(1)}$ has pdf

\[
f_1(x) = \begin{cases} 
    n(1 - x)^{n-1} & 0 < x < 1 \\
    0 & \text{otherwise}
\end{cases}
\]

Now, let us consider the joint distribution of any subset of the order statistics $X_{(1)}$, $\ldots$, $X_{(n)}$. We will first look at the full joint pdf of all of the order statistics.

Pictorially, when $n = 2$, the support of $(X_{(1)}, X_{(2)})^T$ is "half" of the support of $(X_1, X_2)^T$.

In general, when we have $(X_1, \ldots, X_n)^T$, for points $(X_1, \ldots, X_n)^T$ in the support of the distribution of $(X_1, \ldots, X_n)^T$, the components of $(X_1, \ldots, X_n)^T$ can have one of $n!$ orderings. However, points $(X_{(1)}, \ldots, X_{(n)})^T$ in the support of the order statistics...
\((X_{c_1}, \ldots, X_{c_n})^T\) must be in the ordering \(x_{c_1} < x_{c_2} < \ldots < x_{c_n}\). Therefore, the support of \((X_{c_1}, \ldots, X_{c_n})^T\) is \(\frac{1}{n!}\) that of the support of the original \((X_1, \ldots, X_n)^T\).