Joint pdf of Order Statistics

Let \( X_1, \ldots, X_n \) be jointly continuous and i.i.d with common df and pdf \( F \) and \( f \), respectively.

Theorem Let \( f_{1:n}(x_1, \ldots, x_n) \) be the joint pdf of \((X_{(1)}, \ldots, X_{(n)})\)' the vector of order statistics of \( X_1, \ldots, X_n \). Then
\[
f_{1:n}(x_1, \ldots, x_n) = \begin{cases} n! & f(x_1) \cdots f(x_n) \quad \text{for } x_1 < x_2 < \ldots < x_n, \\ 0 & \text{otherwise} \end{cases}
\]

Lemma Suppose \( Y_1, \ldots, Y_n \) are jointly continuous with joint df \( G(y_1, \ldots, y_n) \) and joint pdf \( g(y_1, \ldots, y_n) \). Then
\[
g(y_1, \ldots, y_n) = \lim_{h \to 0} \frac{P(Y_1 \leq y_1 \leq y_1 + h, \ldots, y_n \leq y_n \leq y_n + h)}{h^n}.
\]

Partial Proof (n=1 and n=2 cases)

\[ h=1 \text{ case } \]
g(\( y_1 \)) = \frac{d}{dy_1} G(\( y_1 \)) = \lim_{h \to 0} \frac{G(\( y_1 + h \)) - G(\( y_1 \))}{h}
\]
\[ = \lim_{h \to 0} \frac{P(\( y_1 \leq y_1 + h \)) - P(\( y_1 \leq y_1 \))}{h}
\]
\[ = \lim_{h \to 0} \frac{P(\( y_1 \leq y_1 \leq y_1 + h \))}{h}
\]

\[ h=2 \text{ case } \]
g(\( y_1, y_2 \)) = \frac{d^2}{dy_1 dy_2} G(\( y_1, y_2 \))
\]
\[ = \frac{d}{dy_2} \left[ \lim_{h_1 \to 0} \frac{G(\( y_1 + h_1, y_2 \)) - G(\( y_1, y_2 \))}{h_1} \right]
\]
\[ = \lim_{h_2 \to 0} \left[ \lim_{h_1 \to 0} \frac{G(\( y_1 + h_1, y_2 + h_2 \)) - G(\( y_1, y_2 + h_2 \))}{h_1} \right. 
\]
\[ \quad - \left. \lim_{h_1 \to 0} \frac{G(\( y_1 + h_1, y_2 \)) - G(\( y_1, y_2 \))}{h_1} \right] \frac{1}{h_2}
\]

If we set \( h_1 = h_2 = h \), then we can write this as
\[
\lim_{h \to 0} \frac{G(y_1 + h, y_2 + h) - G(y_1, y_2 + h) - G(y_1 + h, y_2) + G(y_1, y_2)}{h^2}
\]

\[
\lim_{h \to 0} \frac{P(y_1 \leq y_1 \leq y_1 + h, y_2 \leq y_2 \leq y_2 + h)}{h^2}
\]

**Proof of Theorem**

\[
f_{1 \cdots n}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{P(x_1 \leq X_{\sigma(1)} \leq x_1 + h, \ldots, x_n \leq X_{\sigma(n)} \leq x_n + h)}{h^n}
\]

Since \(x_1 < x_2 < \ldots < x_n\), choose \(h\) small enough so that the intervals \([x_1, x_1 + h]\), \ldots, \([x_n, x_n + h]\) are all disjoint. Then

\[
f_{1 \cdots n}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{1}{h^n} P\left( \bigcup_{\sigma} \{x_1 \leq X_{\sigma(1)} \leq x_1 + h, \ldots, x_n \leq X_{\sigma(n)} \leq x_n + h\} \right)
\]

where \(\sigma = (\sigma(1), \ldots, \sigma(n))\) is a permutation of \([1, \ldots, n]^2\), and the union is over all \(n!\) permutations.

Then, since the events in the above union are all mutually disjoint, we have

\[
f_{1 \cdots n}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{1}{h^n} \sum_{\sigma} P(x_1 \leq X_{\sigma(1)} \leq x_1 + h, \ldots, x_n \leq X_{\sigma(n)} \leq x_n + h)
\]

\[
= \lim_{h \to 0} \frac{1}{h^n} \sum_{\sigma} P(X_1 \leq X_{\sigma(1)} \leq x_1 + h) \times \ldots \times P(X_n \leq X_{\sigma(n)} \leq x_n + h)
\]

Since the \(X_i's\) are independent.
\[
\lim_{h \to 0} \frac{1}{h^n} n! \prod_{i=1}^{n} \left[ P(X_i \leq X_i \leq X_i + h) \right] 
= \lim_{h \to 0} \frac{1}{h^n} n! \prod_{i=1}^{n} \left[ \frac{F(X_i + h) - F(X_i)}{h} \right] 
= n! \prod_{i=1}^{n} \left[ \lim_{h \to 0} \frac{F(X_i + h) - F(X_i)}{h} \right] 
= n! \prod_{i=1}^{n} f(X_i)
\]

( since the $X_i$'s are identically distributed every term in the sum is the same).

\[
= \lim_{h \to 0} \frac{1}{h^n} n! \prod_{i=1}^{n} \left( F(X_i + h) - F(X_i) \right) 
= n! \lim_{h \to 0} \prod_{i=1}^{n} \left( F(X_i + h) - F(X_i) \right)
\]