The final special case of computing marginal pdfs of the order-statistics is the joint marginal pdf of any pair \((X_k, X_r)^T\), where \(r \neq k+1\). Suppose \(k < r\) and \(r \neq k+1\). Start with the joint pdf \(f_{X_k, \ldots, X_r}(X_k, \ldots, X_r)\) of \((X_k, \ldots, X_r)^T\) in \((*)\) from last time. We will integrate out the variables \(X_{k+1}, \ldots, X_{r-1}\) in that order (in total we are integrating out \(r-k-1\) variables).

Integrating out \(X_{k+1}\) gives

\[
f_{X_k, X_{k+2}, \ldots, X_r}(X_k, X_{k+2}, \ldots, X_r) = (\text{stuff } 1) \times (\text{stuff } 2) \times f(X_k) f(X_{k+2}) \ldots f(X_r) \times \int f(X_{k+1}) \, dX_{k+1}
\]

\[
= (\text{stuff } 1) \times (\text{stuff } 2) \times f(X_k) (F(X_{k+2}) - F(X_k)) \times f(X_{k+2}) \ldots f(X_r)
\]

\(X_k < X_{k+2} < X_{k+3} < \ldots < X_r\)

Next, integrating out \(X_{k+2}\) will give

\[
f_{X_k, X_{k+3}, \ldots, X_r}(X_k, X_{k+3}, \ldots, X_r) = (\text{stuff } 1) \times (\text{stuff } 2) \times f(X_k) \left(\frac{F(X_{k+3}) - F(X_k)}{2}\right)^2 \times f(X_{k+3}) \ldots f(X_r)
\]

\(X_k < X_{k+3} < X_{k+4} < \ldots < X_r\)

After the \(r-k-1\) integrations, one will obtain

\[f_{X_k, X_r}(X_k, X_r) = (\text{stuff } 1) \times (\text{stuff } 2) \times f(X_k) \left(\frac{F(X_r) - F(X_k)}{r-k-1}\right)^{r-k-1} f(X_r)
\]

So

\[
f_{X_k, X_r}(X_k, X_r) = \frac{n!}{(r-k-1)!} \frac{F(X_k)^{k-1} (1 - F(X_r))^{n-r}}{(n-r)!} \times f(X_k) f(X_r) \left(\frac{F(X_r) - F(X_k)}{r-k-1}\right)^{r-k-1}
\]

**Example** If \(k=1\) and \(r=n\) we get the joint pdf of \((X_1, X_n)^T\)

\[
f_{1n}(X_1, X_n) = n! \times f(X_1) f(X_n) \left(\frac{F(X_n) - F(X_1)}{n-2}\right)^{n-2} \times X_1 < X_n
\]

\[= n(n-1) f(X_1) f(X_n) (F(X_n) - F(X_1))^{n-2} \quad X_1 < X_n\]
Example 1: If $X_1, \ldots, X_n$ are i.i.d. $\text{Uniform}(0, 1)$ then

\[
    f(x) = \begin{cases} 
        1 & 0 < x < 1 \\
        0 & \text{otherwise}
    \end{cases} \\
    F(x) = \begin{cases} 
        x & \text{if } 0 < x < 1 \\
        1 & \text{if } x \geq 1 \\
        0 & \text{if } x \leq 0
    \end{cases}
\]

Plugging into the previous example we get

\[
    f_{\text{in}}(x_1, x_n) = \begin{cases} 
        n(n-1)(x_n-x_1)^{n-2} & \text{for } 0 < x_1 < x_n < 1 \\
        0 & \text{otherwise}
    \end{cases}
\]

Example 2: Let $X_1, \ldots, X_n$ be i.i.d. $\text{Uniform}(0, 1)$. The statistic $X_{(n)} - X_{(1)}$ is called the range of $\{X_1, \ldots, X_n\}$.

Compute $P(X_{(n)} - X_{(1)} \leq t)$ for $t \in (0, 1)$.

From the previous example the joint pdf of $(X_{(1)}, X_{(n)})^T$ is

\[
    f_{\text{in}}(x_1, x_n) = \begin{cases} 
        n(n-1)(x_n-x_1)^{n-2} & \text{for } 0 < x_1 < x_n < 1 \\
        0 & \text{otherwise}
    \end{cases}
\]

$P(X_{(n)} - X_{(1)} \leq t)$ is computed by integrating $f_{\text{in}}(x_1, x_n)$ over the 2-dimensional region $\{(x_1, x_n) \in \mathbb{R}^2 : 0 < x_1 < x_n < 1 \text{ and } x_n - x_1 \leq t\}$

Drawing this region we have the region in red:

![Red and blue regions](image)

It will be simpler to integrate over the region in blue, and subtract the integral from 1. Then we get

\[
    1 - P(X_{(n)} - X_{(1)} \leq t) = \int_{t}^{1} \int_{0}^{x_1-t} n(n-1)(x_n-x_1)^{n-2} \, dx_1 \, dx_n
\]

\[
    = n(n-1) \int_{t}^{1} \frac{-(x_n-x_1)^{n-1}}{n-1} \bigg|_{0}^{x_n-t} \, dx_n
\]
= n \int_t^{x_n} (x_n^{n-1} - t^{n-1}) \, dx_n \\
= n \left[ \frac{x_n^n}{n} - t^{n-1} x_n \right]_t^{x_n} \\
= n \left( \frac{1}{n} - t^{n-1} \right) - \left( \frac{t^n}{n} - t^{n} \right) \\
= 1 - n t^{n-1} - t^n + n t^n \\
Then \quad P(X_{(n)} - X_{(1)} \leq t) = n t^{n-1} + t^n - n t^n \\
= n t^{n-1} - (n-1) t^n \\
This gives the df of the range for \( t \in (0,1) \). Differentiating this with respect to \( t \) gives the pdf of the range as \\
\[ n (n-1) t^{n-2} - n (n-1) t^{n-1} = n (n-1) t^{n-2} (1-t) \] for \( 0 < t < 1 \).