An important property of the Gamma distribution is that if $X_1 \sim \text{Gamma}(r_1, \lambda)$ and $X_2 \sim \text{Gamma}(r_2, \lambda)$ ($\lambda$ is same for both $X_1$ and $X_2$ but $r_1$ and $r_2$ can be different), and $X_1$ and $X_2$ are independent, then $X_1 + X_2 \sim \text{Gamma}(r_1 + r_2, \lambda)$. Let's show this.

Let $Y_1 = X_1 + X_2$

Let $Y_2 = \frac{X_1}{X_1 + X_2}$

The support of the distribution of $(Y_1, Y_2)^T$ is

$S_4 = \{ (y_1, y_2) \in \mathbb{R}^2 : y_1 > 0 \text{ and } 0 < y_2 < 1 \}$.

The inverse mapping is

$X_1 = Y_1 Y_2$

$X_2 = Y_1 (1 - Y_2)$

The Jacobian is

$J = \begin{bmatrix} Y_2 & Y_1 \\ -Y_1 Y_2 - Y_1 (1 - Y_2) & Y_1 \\ \end{bmatrix}$

$\text{det} J = -Y_1 Y_2 - Y_1 (1 - Y_2) = -Y_1$

So $|J| = Y_1$. Then the joint pdf of $(Y_1, Y_2)^T$ is (for $(y_1, y_2) \in S_4$),

$f_{Y_1,Y_2}(y_1, y_2) = f_X(Y_1 Y_2, Y_1 (1 - Y_2)) Y_1$

$= f_{X_1}(Y_1 Y_2) f_{X_2}(Y_1 (1 - Y_2)) Y_1$

$= \frac{\lambda^{r_1}}{\Gamma(r_1)} (Y_1 Y_2)^{r_1 - 1} \exp(-\lambda Y_1 Y_2) \frac{\lambda^{r_2}}{\Gamma(r_2)} (Y_1 (1 - Y_2))^{r_2 - 1} \exp(-\lambda Y_1 (1 - Y_2)) Y_1$

$= \frac{\lambda^{r_1 + r_2}}{\Gamma(r_1) \Gamma(r_2)} Y_1^{r_1 + r_2 - 1} \exp(-\lambda Y_1) \left[ \frac{\Gamma(r_1 + r_2)}{\Gamma(r_1) \Gamma(r_2)} Y_1^{r_1 - 1} (1 - Y_2)^{r_2 - 1} \right]$

$= \left[ \frac{\lambda^{r_1 + r_2}}{\Gamma(r_1 + r_2)} Y_1^{r_1 + r_2 - 1} \exp(-\lambda Y_1) \right] \left[ \frac{\Gamma(r_1 + r_2)}{\Gamma(r_1) \Gamma(r_2)} Y_1^{r_1 - 1} (1 - Y_2)^{r_2 - 1} \right]$
From the final expression for \( f_{Y_2}(y_1, y_2) \) we can deduce that 
\( Y_1 \) and \( Y_2 \) are independent, \( Y_1 \sim \Gamma(r_1+r_2, \lambda) \), and \( Y_2 \) has pdf 
\[
f_{Y_2}(y_2) = \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)} y_2^{r_1-1} (1-y_2)^{r_2-1} \text{ for } 0 < y_2 < 1 \text{ and } f_{Y_2}(y_2) = 0 \text{ for } y_2 \leq 0 \text{ or } y_2 \geq 1. \]  
(The distribution of \( Y_2 \) is called the Beta distribution with parameters \( r_1 \) and \( r_2 \), we will discuss the Beta distribution more later.)

We can extend the property that \( X_1 + X_2 \sim \Gamma(r_1+r_2, \lambda) \) to any finite number of independent Gamma random variables:

Let \( X_1, \ldots, X_n \) be independent with \( X_i \sim \Gamma(r_i, \lambda) \), \( i = 1, \ldots, n \). Then \( X_1 + X_2 \sim \Gamma(r_1+r_2, \lambda) \). Then applying the result to \( X_1 + X_2 \) and \( X_3 \) we get \( (X_1 + X_2) + X_3 \sim \Gamma(r_1+r_2+r_3, \lambda) \).

If we keep applying the result we end up with 
\( X_1 + X_2 + \ldots + X_n \sim \Gamma(r_1+r_2+\ldots+r_n, \lambda) \).

Two important applications of this are:
1. If \( r_1 = r_2 = \ldots = r_n = 1 \), then \( X_1, \ldots, X_n \) are i.i.d. Exponential(\( \lambda \)), and \( X_1 + \ldots + X_n \sim \Gamma(n, \lambda) \).
2. If \( X_1, \ldots, X_n \) are i.i.d. \( N(0, 1) \) random variables, then \( X_1^2 + \ldots + X_n^2 \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right) \), which is a \( \chi^2_n \) distribution.

We can conclude from this that the sum of independent \( \chi^2 \) random variables has a \( \chi^2 \) distribution with degrees of freedom equal to the sum of the degrees of freedoms of the individual \( \chi^2 \) random variables we are summing.
Example. Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be random points in the plane, where \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\) are i.i.d. \(N(0, \sigma^2)\). What is the probability that the disk of radius 2 centered at the origin does not contain any of these points?

Sol. We wish to compute

\[
P(\min(X_1^2 + Y_1^2, \ldots, X_n^2 + Y_n^2) > 4)
\]

we have that \(\frac{X_i}{\sigma} \sim N(0, 1)\) and \(\frac{Y_i}{\sigma} \sim N(0, 1)\) for \(i = 1, \ldots, n\).

So \(\frac{X_i^2 + Y_i^2}{\sigma^2} \sim \chi^2_2\) for \(i = 1, \ldots, n\) and \(\frac{X_i^2 + Y_i^2}{\sigma^2}, \ldots, \frac{X_n^2 + Y_n^2}{\sigma^2}\)

are independent. Then

\[
P(\min(X_1^2 + Y_1^2, \ldots, X_n^2 + Y_n^2) > 4)
= P(\min\left(\frac{X_1^2 + Y_1^2}{\sigma^2}, \ldots, \frac{X_n^2 + Y_n^2}{\sigma^2}\right) > \frac{4}{\sigma^2})
\]

Next note that the \(\chi^2_2\) distribution is the Gamma \((1, \frac{1}{2})\), which is the Exponential \((\frac{1}{2})\) distribution.

The pdf of \(\min\left(\frac{X_1^2 + Y_1^2}{\sigma^2}, \ldots, \frac{X_n^2 + Y_n^2}{\sigma^2}\right)\) is then

\[
n \frac{1}{2} e^{-\frac{1}{2}x} \left(1 - (1 - e^{-\frac{1}{2}x})^{n-1}\right) = \frac{n}{2} e^{-\frac{1}{2}x}, \quad \text{i.e., the Exponential \((\frac{n}{2})\) density}.
\]

So the desired probability is

\[
e^{-\frac{n}{2} \frac{4}{\sigma^2}} = e^{-2n/\sigma^2}.
\]