Example: Say we have \( n_i \) objects of type \( i \), \( i = 1, \ldots, r \). Let \( N = n_1 + \ldots + n_r \) be the total number of objects. Suppose we draw a sample of size \( n \leq N \) with replacement (if the sample is drawn without replacement then this is the experiment giving rise to the Multivariate Hypergeometric distribution). If \( X_i = \# \) of objects of type \( i \) in the sample, \( i = 1, \ldots, r \), then \( X = (X_1, \ldots, X_r)^T \) has a Multinomial distribution with parameters \( n \) and \( p_i = \frac{n_i}{N} \), \( i = 1, \ldots, r \).

Alternative Description of the Multinomial Distribution

We can describe the distribution with a joint pmf on \( \mathbb{R}^{r-1} \) because of the constraint that \( X_1 + \ldots + X_r = n \). We can let \( X = (X_1, \ldots, X_{r-1})^T \). Then \( P(X_1 = x_1, \ldots, X_{r-1} = x_{r-1}, X_r = n - x_1 - \ldots - x_{r-1}) \)

\[
= \frac{n!}{x_1! (n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1} \]

for \( (x_1, \ldots, x_{r-1}) \in \{(x_1, \ldots, x_{r-1}) \in \mathbb{R}^{r-1} ; x_i \in \{0,1,\ldots,n\}, i = 1, \ldots, r-1, \text{ and } x_1 + \ldots + x_{r-1} \leq n\} \)

If \( r=2 \), we get the usual form of the Binomial \((n, p)\), where \( p = p_1 = \) probability of outcome \( 1 \):

\[
P(X_1 = x_1) = \binom{n}{x_1} p_1^{x_1} (1-p_1)^{n-x_1} = \frac{n!}{x_1! (n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1} \]

for \( x_1 = 0, 1, \ldots, n \).

Marginal Distributions

Let \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) with \( k < n \). We wish to find the marginal joint distribution of \((X_{i_1}, \ldots, X_{i_k})^T\). Relabel the outcomes in each multinomial experiment that are not one of the outcome \( i_1, \ldots, i_k \) as outcome \( 0 \). Then each multinomial experiment will have \( k+1 \) possible outcomes: \( i_1, \ldots, i_k \) and 0, with probabilities \( p_{i_1}, \ldots, p_{i_k} \), and \( 1 - p_{i_1} - \ldots - p_{i_k} \).

Then 
\[ P(X_{i_1} = x_{i_1}, \ldots, X_{i_k} = x_{i_k}) \]
\[ = P(X_{i_1} = x_{i_1}, \ldots, X_{i_k} = x_{i_k}, X_0 = n - x_{i_1} - \ldots - x_{i_k}) \]
\[ = \prod_{i=1}^{k} p_{x_{i_1}} \prod_{i=1}^{k} p_{x_{i_k}} (1 - p_{x_1} - \ldots - p_{x_k})^{n - x_{i_1} - \ldots - x_{i_k}} \]

For \((x_{i_1}, \ldots, x_{i_k}) \in \{(x_{i_1}, \ldots, x_{i_k}) \in \mathbb{R}^k : x_j \in \{0, 1, \ldots, n\}, j = 1, \ldots, k \text{ and } x_{i_1} + \ldots + x_{i_k} \leq n \}, \)

If \((x_{i_1}, \ldots, x_{i_k})\) is not in the above support then 
\[ P(X_{i_1} = x_{i_1}, \ldots, X_{i_k} = x_{i_k}) = 0 \]

This gives the joint pmf of \((X_{i_1}, \ldots, X_{i_k})^\top\).

If \(k=1\), then \(X_{i_1}\) has a Binomial \((n, p_{i_1})\).

**Example**

Suppose 5 cards are drawn with replacement from an ordinary deck of 52 cards. What is the probability that the set of 5 cards drawn has exactly one 2 of diamonds, 3 clubs, and 2 red cards.

**Sol.** Let outcome 1 = draw a 2 of diamonds

outcome 2 = draw a red card that is not a 2 of diamonds

outcome 3 = draw a club

outcome 4 = draw a spade

Let \(X_i = \#\) of draws (out of 5 draws) in which outcome \(i\) occurred, \(i = 1, \ldots, 4\).

I want 
\[ P(X_1 = 1, X_2 = 1, X_3 = 3) \]
\[ = P(X_1 = 1, X_2 = 1, X_3 = 3, X_4 = 0) \]
\((X_1, X_2, X_3, X_4)\) has a Multinomial Distribution with parameters \(n=5\), \(p_1 = \frac{1}{52}\), \(p_2 = \frac{25}{52}\), \(p_3 = \frac{1}{4}\), \(p_4 = \frac{1}{4}\).
Then
\[
P(X_1 = 1, X_2 = 1, X_3 = 3) = \binom{5}{1, 3, 0} \left( \frac{1}{52} \right)^1 \left( \frac{25}{52} \right)^1 \left( \frac{1}{4} \right)^3 \left( \frac{1}{4} \right)^0
\]
\[
= \frac{5!}{1! \cdot 1! \cdot 3!} \frac{25}{52 \times 52 \times 64}
\]
\[
= \frac{20 \times 25}{52 \times 52 \times 64} \approx 0.00289
\]