Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination   February 13, 2015

- Total points = 30. Duration = 58 minutes.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.
- The last page contains formulas you may find useful. Please check this page first.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Total: [30] ____________

1. A family of Yellow-Faced (YF) gophers consisting of 2 parents and 3 children are kept in a laboratory. In addition to these a family of YF gophers with 2 parents and 4 children, a family of Big Pocket (BP) gophers with 2 parents and 5 children, and a family of BP gophers with 1 mother and 4 children are also kept in the laboratory. A sample of 4 gophers is selected at random from among all the gophers in the laboratory. What is the probability that the sample consists of one adult female, one adult male, and 2 children, with both adults of the same genus (either both YF or both BP). [10]

Solution: Let X_1 be the number of male YF gophers, X_2 the number of female YF gophers, X_3 the number of male BP gophers, X_4 the number of female BP gophers, and X_5 the number of child gophers in the sample. The sample size is $n = 4$ and the total number of gophers in the laboratory is 23. The total numbers of YF male, YF female, BP male, BP female, and children gophers are 2, 2, 1, 2 and 16 respectively. The joint distribution of $(X_1, X_2, X_3, X_4, X_5)^T$ is Multivariate Hypergeometric and the desired probability is

$$\begin{aligned}
&P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 0, X_5 = 2) + P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 1, X_5 = 2) \\
&\quad = \frac{\binom{2}{1}\binom{2}{1}\binom{1}{0}\binom{2}{0}\binom{16}{2}}{\binom{23}{4}} + \frac{\binom{2}{0}\binom{2}{0}\binom{1}{1}\binom{2}{1}\binom{16}{2}}{\binom{23}{4}} \\
&\quad = \frac{2 \times 2 \times 1 \times 1 \times 120}{8855} + \frac{1 \times 1 \times 1 \times 2 \times 120}{8855} = \frac{720}{8855} \approx 0.0813.
\end{aligned}$$

2. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent Uniform(0,1) random variables. We form n rectangles, where the ith rectangle has adjacent sides of length X_i and Y_i, for $i = 1, \ldots, n$. Let A_i be the area of the ith rectangle, $i = 1, \ldots, n$, and define $A_{\max} = \max(A_1, \ldots, A_n)$. Find the pdf of $A_{\max}$. [10]

Solution: First note that $A_i = X_i Y_i$ for $i = 1, \ldots, n$, and $A_1, \ldots, A_n$ are independent since the vectors $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent. Also, the A_i are identically distributed, each with support $(0, 1)$. We first find $P(A_i \le a)$ for any $a \in (0, 1)$, which is $P(X_i Y_i \le a)$. If $Y_i \le a$ then the limits for X_i are from 0 to 1. If $Y_i > a$ then the limits for X_i are from 0 to a/Y_i. The joint pdf of (X_i, Y_i) is $f(x, y) = 1$ for $0 < x, y < 1$. Thus we have

$$\begin{aligned} P(A_i \le a) &= P(X_i Y_i \le a) = \int_0^a \int_0^1 dxdy + \int_a^1 \int_0^{a/y} dxdy \\ &= a + a \int_a^1 \frac{1}{y} dy = a + a(\ln 1 - \ln a) = a(1 - \ln a). \end{aligned}$$

Next, we get the cdf of $A_{\max}$ as

$$F(a) = P(A_{\max} \le a) = P(A_1 \le a, \ldots, A_n \le a) = P(A_1 \le a)^n = (a(1 - \ln a))^n,$$

for $0 < a < 1$. Finally, we obtain the pdf f of $A_{\max}$ by differentiation:

$$f(a) = F'(a) = n(a(1 - \ln a))^{n-1} [1 - \ln a - 1] = -n \ln a (a(1 - \ln a))^{n-1},$$

which is valid for $a \in (0, 1)$, and $f(a) = 0$ for $a \le 0$ or $a \ge 0$.

3. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{y^3}{2}e^{-y(x+1)} & \text{for } x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal pdf of X. [5]

Solution: Integrating the joint pdf over y gives the marginal pdf of X. For $x > 0$, we have

$$f_X(x) = \int_0^\infty \frac{y^3}{2}e^{-y(x+1)}dy = \frac{\Gamma(4)}{2(x+1)^4}\int_0^\infty \frac{(x+1)^4}{\Gamma(4)}y^3 e^{-y(x+1)}dy = \frac{3}{(x+1)^4},$$

as the integrand in the last integral over y is a Gamma$(4, x+1)$ pdf, and so integrates (over y) to 1. For $x \le 0$, $f_X(x) = 0$.

(b) Find the marginal pdf of Y and $E[Y]$. [5]

Solution: Integrating the joint pdf over x gives the marginal pdf of Y. For $y > 0$, we have

$$f_Y(y) = \int_0^\infty \frac{y^3}{2}e^{-y(x+1)}dx = \frac{y^2}{2}e^{-y}\int_0^\infty ye^{-yx}dx = \frac{y^2}{2}e^{-y},$$

as the integrand in the last integral over x is an Exponential(y) pdf, and so integrates (over x) to 1. For $y \le 0$, $f_Y(y) = 0$. We can recognize the marginal distribution of Y to be Gamma$(3, 1)$, and so $E[Y] = 3/1 = 3$.

Formulas:

- The Uniform(0,1) distribution has pdf

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$