Surname ____________ Given Name ____________ Student Number ____________

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination February 17, 2017

- Total points = 30. Duration = 58 minutes.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.
- The last page contains formulas you may find useful. Please check this page first.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Total: [30] ____________

1. (a) Let X_1, X_2, X_3 be independent $N(0,1)$ random variables. Find the probability density function of $U = X_1^2 + X_2^2 + X_3^2$. [5]

(b) Suppose that the random vector $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ is uniformly distributed on the sphere of radius 1 centred at the origin; that is, $\boldsymbol{Y}$ has joint probability density function

$$f_{\boldsymbol{Y}}(y_1, y_2, y_3) = \begin{cases} \frac{3}{4\pi} & \text{if } (y_1, y_2, y_3) \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 \leq 1\}$ is the sphere of radius 1 centred at $(0,0,0)$. Let $V = Y_1^2 + Y_2^2 + Y_3^2$. Find the probability density function of V and find $E[V]$. [5]

2. Let $X_1, \ldots, X_n$ be a sequence of independent random variables uniformly distributed on the interval $(0,1)$, and let $X_{(1)}, \ldots, X_{(n)}$ denote their order statistics. For fixed k let $g_n(x)$ denote the probability density function of $nX_{(k)}$. Find $g_n(x)$ and show that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \frac{x^{k-1}}{(k-1)!}e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

which is the Gamma(k,1) density. [10]

3. Let $X_1, \ldots, X_6$ be independent random variables uniformly distributed on the interval $(0, 1)$. Find the pdf of $U = \min\{\max(X_1, X_2), \max(X_3, X_4), \max(X_5, X_6)\}$. [10]

Formulas:

- The volume of a sphere of radius r is $\dfrac{4}{3}\pi r^3$.
- Uniform distribution on the interval $(0, 1)$ has pdf

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- The standard normal distribution has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{for } x \in \mathbb{R}$$

- Beta density with parameters α and β:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Gamma density with parameters r and λ:

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$