Surname Given Name Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination Solutions February 17, 2017

- Total points = 30. Duration = 58 minutes.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.
- The last page contains formulas you may find useful. Please check this page first.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Total: [30] ____________

1. (a) Let X_1, X_2, X_3 be independent $N(0,1)$ random variables. Find the probability density function of $U = X_1^2 + X_2^2 + X_3^2$. [5]

Solution: We know from class that X_i^2 has the χ_1^2 distribution,i.e., the Gamma$(1/2, 1/2)$ distribution. Since X_1, X_2, X_3 are independent, so are X_1^2, X_2^2, X_3^2, so $U = X_1^2 + X_2^2 + X_3^2$ is a sum of 3 independent Gamma$(1/2, 1/2)$ random variables, and as such has the Gamma$(3/2, 1/2)$ distribution. Thus

$$f_U(u) = \begin{cases} \frac{(1/2)^{3/2}}{\Gamma(3/2)} u^{1/2} e^{-(1/2)u} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The expression can be simplified by noting that $\Gamma(3/2) = (1/2)\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$.

(b) Suppose that the random vector $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ is uniformly distributed on the sphere of radius 1 centred at the origin; that is, $\boldsymbol{Y}$ has joint probability density function

$$f_{\boldsymbol{Y}}(y_1, y_2, y_3) = \begin{cases} \frac{3}{4\pi} & \text{if } (y_1, y_2, y_3) \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 \le 1\}$ is the sphere of radius 1 centred at $(0,0,0)$. Let $V = Y_1^2 + Y_2^2 + Y_3^2$. Find the probability density function of V and find $E[V]$. [5]

Solution: We can find the pdf of V by first computing the cdf of V. For $v \in [0,1]$ we have

$$F_V(v) = P(V \le v) = P(X_1^2 + X_2^2 + X_3^2 \le (\sqrt{v})^2) = P((X_1, X_2, X_3) \in S_{\sqrt{v}}),$$

where $S_{\sqrt{v}}$ is the sphere of radius $\sqrt{v}$. Thus, for $v \in [0,1]$

$$\begin{aligned} F_V(v) &= \iiint\limits_{S_{\sqrt{v}}} f_X(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \iiint\limits_{S_{\sqrt{v}}} \frac{3}{4\pi} dx_1 dx_2 dx_3 \\ &= \frac{3}{4\pi} \times \text{Volume of } S_{\sqrt{v}} \\ &= \frac{3}{4\pi} \times \frac{4\pi(\sqrt{v})^3}{3} = v^{3/2}. \end{aligned}$$

We also have $F_V(v) = 0$ for $v < 0$ and $F_V(v) = 1$ for $v > 1$. Differentiating, we obtain the pdf of V as

$$f_V(v) = F_V'(v) = \begin{cases} \frac{3\sqrt{v}}{2} & \text{for } 0 \le v \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E[V] = \frac{3}{2} \int_0^1 v^{3/2} dv = \frac{3}{2} \left[\frac{2}{5} v^{5/2} \right]_0^1 = \frac{3}{5}.$$

2. Let $X_1, \ldots, X_n$ be a sequence of independent random variables uniformly distributed on the interval $(0,1)$, and let $X_{(1)}, \ldots, X_{(n)}$ denote their order statistics. For fixed k let $g_n(x)$ denote the probability density function of $nX_{(k)}$. Find $g_n(x)$ and show that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \frac{x^{k-1}}{(k-1)!}e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

which is the Gamma(k,1) density. [10]

<u>*Solution:*</u> The pdf of $X_{(k)}$ is

$$f_k(x_k) = \begin{cases} \frac{n!}{(k-1)!(n-k)!} x_k^{k-1}(1-x_k)^{n-k} & \text{for } 0 < x_k < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = nX_{(k)}$. Then the set of possible values of X (the sample space of X) is $\{x : 0 < x < n\}$. So for $x \in (0,n)$, by the (1-dimensional) change of variable formula the pdf of X is given by

$$g_n(x) = f_k(x/n)\frac{1}{n} = \frac{(n-1)!}{(k-1)!(n-k)!}\left(\frac{x}{n}\right)^{k-1}\left(1-\frac{x}{n}\right)^{n-k}$$

and $g_n(x) = 0$ for $x \notin (0,n)$. Now fix $x > 0$ and let $n > x$. Then

$$\begin{aligned} g_n(x) &= \frac{(n-1)!}{(k-1)!(n-k)!}\left(\frac{x}{n}\right)^{k-1}\left(1-\frac{x}{n}\right)^{n-k} \\ &= \frac{(n-1)\times\ldots\times(n-k+1)}{n^{k-1}}\left(1-\frac{x}{n}\right)^{-k}\frac{1}{(k-1)!}x^{k-1}\left(1-\frac{x}{n}\right)^{n} \\ &\to \frac{1}{(k-1)!}x^{k-1}e^{-x}, \end{aligned}$$

as desired, since

$$\begin{aligned} \frac{(n-1)\times\ldots\times(n-k+1)}{n^{k-1}} &= \left(\frac{n-1}{n}\right)\times\ldots\times\left(\frac{n-k+1}{n}\right) \\ &\to 1\times\ldots\times 1 = 1, \end{aligned}$$

$(1-x/n)^{-k}$ clearly converges to 1 as $n\to\infty$ (for fixed k), and $(1-x/n)^n \to e^{-x}$ as $n\to\infty$ (from calculus). Since $g_n(x) = 0$ for $x < 0$ it is obvious that $g_n(x)\to 0$ as $n\to\infty$ if $x < 0$.

3. Let $X_1, \ldots, X_6$ be independent random variables uniformly distributed on the interval $(0,1)$. Find the pdf of $U = \min\{\max(X_1, X_2), \max(X_3, X_4), \max(X_5, X_6)\}$. [10]

Solution: Let $Y_1 = \max(X_1, X_2)$, $Y_2 = \max(X_3, X_4)$, and $Y_3 = \max(X_5, X_6)$. Then Y_1, Y_2, Y_3 are independent and identically distributed random variables, each with cdf

$$F_Y(y) = P(X_1 \le y, X_2 \le y) = [P(X_1 \le y)]^2 = \begin{cases} y^2 & \text{for } 0 < y < 1 \\ 0 & \text{for } y \le 0 \\ 1 & \text{for } y \ge 1. \end{cases}$$

and pdf

$$f_Y(y) = \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $U = \min(Y_1, Y_2, Y_3)$ has cdf

$$\begin{aligned} F_U(u) &= 1 - P(\min(Y_1, Y_2, Y_3) > u) \\ &= 1 - P(Y_1 > u, Y_2 > u, Y_3 > u) \\ &= 1 - (1 - F_Y(u))^3 \\ &= \begin{cases} 1 - (1 - u^2)^3 & \text{for } 0 < u < 1 \\ 0 & \text{for } u \le 0 \\ 1 & \text{for } u \ge 1. \end{cases} \end{aligned}$$

By differentiation, then, U has pdf given by

$$f_U(u) = \begin{cases} 6u(1 - u^2)^2 & \text{for } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Formulas:

- The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
- Uniform distribution on the interval $(0, 1)$ has pdf

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- The standard normal distribution has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{for } x \in \mathbb{R}$$

- Beta density with parameters α and β:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Gamma density with parameters r and λ:

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$