Student Number

Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination March 2, 2018

- Total points = 30. Duration = 60 minutes.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- Write the answers in the space provided, continue on the backs of pages if needed.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
- Marks per part question are shown in brackets at the right margin.
- The last page contains formulas you may find useful. Please check this page first.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Total: [30] ____________

1. Let X_{ij}, $i = 1, 2, 3$ and $j = 1, 2, 3$, be mutually independent Uniform$(0, 1)$ random variables. Let $X_{(2),j}$ be the sample median of X_{1j}, X_{2j}, X_{3j}, for $j = 1, 2, 3$. Find the probability that exactly one of $X_{(2),1}, X_{(2),2}$, and $X_{(2),3}$ is in the interval $[0, 1/3)$, exactly one is in $[1/3, 2/3)$, and exactly one is in $[2/3, 1)$. [10]

Solution: The pdf of $X_{(2),j}$, which is the second order statistic of X_{1j}, X_{2j}, and X_{3j} is given by

$$f_2(x) = \begin{cases} 6x(1-x) & \text{for } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Note that $X_{(2),1}, X_{(2),2}$, and $X_{(2),3}$ are independent and identically distributed. The cdf of $X_{(2),j}$ is then

$$F_2(x) = \begin{cases} 3x^2 - 2x^3 & \text{for } x \in [0,1] \\ 0 & \text{for } x < 0 \\ 1 & \text{for } x > 1. \end{cases}$$

Then

$$\begin{aligned} P(X_{(2),j} \in [0, 1/3)) &= F_2(1/3) = \frac{3}{9} - \frac{2}{27} = \frac{7}{27} \\ P(X_{(2),j} \in [1/3, 2/3)) &= F_2(2/3) - F_2(1/3) = \frac{4}{3} - \frac{16}{27} - \frac{7}{27} = \frac{13}{27} \\ P(X_{(2),j} \in [2/3, 1)) &= F_2(1) - F_2(2/3) = 1 - \frac{20}{27} = \frac{7}{27}. \end{aligned}$$

Finally, if we let N_1 be the number of the sample medians that fall in the interval $[0, 1/3)$, N_2 the number that fall in $[1/3, 2/3)$, and N_3 the number that fall in $[2/3, 1)$, then (N_1, N_2, N_3) will follow a Multinomial distribution with parameters $n = 3$, $p_1 = 7/27$, $p_2 = 13/27$, and $p_3 = 7/27$. The probability we want is

$$\begin{aligned} P(N_1 = 1, N_2 = 1, N_3 = 1) &= \frac{3!}{1!1!1!} p_1 p_2 p_3 \\ &= 6 \times \frac{7}{27} \times \frac{13}{27} \times \frac{7}{27} \\ &= \frac{1274}{6561} \approx 0.194. \end{aligned}$$

2. Let X and Y be jointly continuous random variables with joint pdf

$$f(x,y) = \begin{cases} 3|x|y & \text{for } -1 \le x \le 1 \text{ and } x^2 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_X(x)$ and $f_Y(y)$, the marginal pdfs of X and Y, respectively, as well as $E[X]$ and $E[Y]$. [10]

Solution: For $x \in [-1,1]$, the marginal pdf of X is given by

$$f_X(x) = \int_{x^2}^{1} 3|x|y dy = 3|x|\frac{y^2}{2}\Big|_{x^2}^{1} = \frac{3}{2}|x|(1-x^4).$$

For $x \notin [-1,1]$, $f_X(x) = 0$. For $y \in [0,1]$ the marginal pdf of Y is

$$f_Y(y) = 3y\int_{-\sqrt{y}}^{\sqrt{y}} |x|dx = 6y\int_0^{\sqrt{y}} xdx = 6y\frac{x^2}{2}\Big|_0^{\sqrt{y}} = 3y^2.$$

For $y \notin [0,1]$, $f_Y(y) = 0$. $E[X] = 0$ since the pdf of X is symmetric about 0 and the integral is finite. Finally, for $E[Y]$ we have

$$E[Y] = 3\int_0^1 y^3 dy = 3\frac{y^4}{4}\Big|_0^1 = \frac{3}{4}.$$

3. Suppose an urn contains 52 balls and suppose that 4 of the balls are numbered "1", 4 are numbered "2", ..., 4 are numbered "13" (i.e., exactly 4 of the balls are numbered "i", for $i = 1, \ldots, 13$). Suppose all 52 balls are drawn at random, one at a time, without replacement, and the balls are paired up as they are drawn to form 26 pairs (i.e., the ith pair is the two balls drawn on draw $2i - 1$ and draw $2i$, for $i = 1, \ldots, 26$). Find the expected number of pairs that have the same number on both balls. [10]

Solution: Let X be the number of pairs that have the same number on both balls and let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th pair has the same number on both balls} \\ 0 & \text{if the } i\text{th pair has different numbers on the two balls} \end{cases}$$

for $i = 1, \ldots, 26$. Then $X = X_1 + \ldots + X_{26}$ and $E[X] = E[X_1] + \ldots + E[X_{26}]$. We have

$$E[X_i] = P(i\text{th pair has the same number on both balls}).$$

The above probability can be computed by considering all permutations of the balls, which are all equally likely since the drawing is random. There are 52! permutations of the balls. It remains to count the number of permutations that have balls with the same number in positions $2i - 1$ and $2i$. This is a counting task. First, choose a number. There are 13 ways to do this. Next, pick two balls with the chosen number. There are $\binom{4}{2} = 6$ ways to do this. Next, place the chosen balls in positions $2i - 1$ and $2i$. There are 2 ways to do this. Lastly, place the remaining 50 balls in the remaining 50 positions. There are 50! ways to do this. Then, the total number of permutations that have balls with the same number in positions $2i - 1$ and $2i$ is $13 \times 6 \times 2 \times 50!$, and

$$E[X_i] = \frac{13 \times 6 \times 2 \times 50!}{52!} = \frac{3}{51}.$$

This is the same for all i. Therefore,

$$E[X] = \frac{26 \times 3}{51} = \frac{78}{51} \approx 1.53.$$

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