MTHE/STAT 353 - MIDTERM EXAM

Thursday Feb.27, 2020

SOLUTIONS

Instructions:

- There are 3 questions, each worth 10 marks, for a total of 30.
- The duration of the exam is 60 minutes.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted. A simple calculator is permitted.
- Write the answers in the space provided.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.

STUDENT NUMBER

1. Let $X = (X_1, X_2)^T$ be uniformly distributed on the positive quadrant intersected with the disk of radius 1 centred at the origin; i.e., X has joint pdf $f_X(x_1, x_2) = \frac{4}{\pi} I_{S_X}(x_1, x_2)$, where

$$S_X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1^2 + x_2^2 \leq 1\}$$

Let $Y_1 = X_1^2$ and $Y_2 = X_2^2$. Find the joint pdf of $(Y_1, Y_2)^T$ and the marginal pdf of Y_1.

Solution: We use the bivariate change of variable formula to get the joint pdf of (Y_1, Y_2). The inverse transformation is $X_1 = Y_1^{1/2}$ and $X_2 = Y_2^{1/2}$. The Jacobian is

$$\mathbf{J} = \det \begin{bmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ 0 & \frac{1}{2\sqrt{y_2}} \end{bmatrix} = \frac{1}{4\sqrt{y_1 y_2}}.$$

The support of (Y_1, Y_2) is

$$S_Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \leq 1\}.$$

By the change of variable formula, the joint pdf of (Y_1, Y_2) is given by

$$f_Y(y_1, y_2) = f_X\left(\frac{1}{2\sqrt{y_1}}, \frac{1}{2\sqrt{y_2}}\right) |\mathbf{J}| = \frac{4}{\pi} \frac{1}{4\sqrt{y_1 y_2}} I_{S_Y}(y_1, y_2) = \frac{1}{\pi\sqrt{y_1 y_2}} I_{S_Y}(y_1, y_2).$$

The marginal pdf of Y_1 is obtained by integration. For $y_1 \in [0, 1]$,

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_Y(y_1, y_2) dy_2 \\ &= \int_0^{1-y_1} \frac{1}{\pi\sqrt{y_1 y_2}} dy_2 \\ &= \frac{1}{\pi\sqrt{y_1}} \left[2\sqrt{y_2}\right]_0^{1-y_1} \\ &= \frac{2\sqrt{1-y_1}}{\pi\sqrt{y_1}}, \end{aligned}$$

and $f_{Y_1}(y_1) = 0$ otherwise.

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2. Suppose we have n disks, where the radius of disk i is X_i, and $X_1, \ldots, X_n$ are iid Uniform(0,1) random variables. Find the expected areas of the largest disk and the second largest disk.

   Solution: The areas of the two largest disks are $\pi X_{(n)}^2$ and $\pi X_{(n-1)}^2$, where $X_{(n-1)}$ and $X_{(n)}$ are the two largest order statistics of $X_1, \ldots, X_n$. The pdfs of $X_{(n)}$ and $X_{(n-1)}$ are, respectively,

$$\begin{aligned} f_n(x_n) &= n x_n^{n-1} I_{[0,1]}(x_n) \\ f_{n-1}(x_{n-1}) &= n(n-1) x_{n-1}^{n-2}(1 - x_{n-1}) I_{[0,1]}(x_{n-1}) \end{aligned}$$

   Then the expected area of the largest disk is

$$E[\pi X_{(n)}^2] = \pi \int_0^1 x^2 n x^{n-1} dx = \pi n \frac{x^{n+2}}{n+2}\bigg|_0^1 = \frac{\pi n}{n+2},$$

   and the expected area of the second largest disk is

$$\begin{aligned} E[\pi X_{(n-1)}^2] &= \pi \int_0^1 x^2 n(n-1) x^{n-2}(1-x) dx \\ &= \pi n(n-1)\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 \\ &= \frac{\pi n(n-1)}{(n+1)(n+2)}. \end{aligned}$$

STUDENT NUMBER

3. Suppose an urn has one red ball, one blue ball, and one green ball. 10 draws are performed. On each draw the drawn ball is returned to the urn. What is the probability that in the 10 draws, at least 7 of the balls drawn were of the same colour and each of the colours was drawn at least once?

   Solution: Let X_1 be the number of draws where a red ball was drawn, X_2 the number of draws where a blue ball was drawn, and X_3 the number of draws where a green ball was drawn. Then (X_1, X_2, X_3) has a Multinomial distribution with parameters $n = 10$ and $p_1 = p_2 = p_3 = 1/3$. Let A denote the event of interest. Then we can decompose A as $A = A_1 \cup A_2 \cup A_3$, where A_1, A_2, A_3 are disjoint and A_i is the subset of A corresponding to $X_i \geq 7$. By symmetry $P(A_1) = P(A_2) = P(A_3)$ so $P(A) = 3P(A_1)$. For $P(A_1)$ we have

$$\begin{aligned} P(A_1) &= P(X_1 = 7, X_2 = 1, X_3 = 2) + P(X_1 = 7, X_2 = 2, X_3 = 1) \\ &\quad + P(X_1 = 8, X_2 = 1, X_3 = 1) \\ &= \left(\frac{1}{3}\right)^{10} \left[\frac{10!}{7!1!2!} + \frac{10!}{7!2!1!} + \frac{10!}{8!1!1!}\right] \\ &= \frac{360 + 360 + 90}{3^{10}} = \frac{810}{3^{10}} = \frac{810}{59049}. \end{aligned}$$

   Then

$$P(A) = \frac{2430}{59049} \approx 0.04115.$$