Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination Solutions 2021

- **THE SOLUTIONS ARE TO BE UPLOADED TO CROWDMARK**. You should have received an invitation from Crowdmark to submit your solutions.
- **Duration = 120 minutes + 30 minutes to prepare your solutions for upload to Crowdmark and to submit your solutions to Crowdmark.**
- Write your solutions using your own paper. A tablet such as an ipad is **not allowed**. Use a dark lead pencil or pen to write your solutions. Solutions which cannot be read because they are too faint will not be marked.
- Write your student number at the start of each solution and number each solution.
- Total points = 30. Each question is worth 10 marks. Marks per part question are shown in brackets at the right.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.

1. Let X_1, X_2, X_3 be independent and identically distributed discrete random variables with common pmf $p_X(x) = \left(\frac{1}{2}\right)^{y+1}$ for $x = 0, 1, 2, \ldots$ and $p_X(x) = 0$ otherwise (the mean of this distribution is 1 and the variance is 2). Consider the points (X_1, X_2) and (X_2, X_3) in the xy-plane.

(a) Find the probability that neither of the points (X_1, X_2) or (X_2, X_3) fall on the line $y = x$. (I want you to write out an appropriate triple sum and evaluate it). [5]

(b) Find the expected squared Euclidean distance between the points (X_1, X_2) and (X_2, X_3). [5]

Solution:

(a) We want $P(\{X_1 \neq X_2\} \cap \{X_2 \neq X_3\})$. By independence, the joint pmf of $(X_1, X_2, X_3)^T$ is $p_X(x_1, x_2, x_3) = \left(\frac{1}{2}\right)^{x_1+1} \left(\frac{1}{2}\right)^{x_2+1} \left(\frac{1}{2}\right)^{x_3+1}$ for $x_1, x_2, x_3 = 0, 1, 2, \ldots$ and $p_X(x_1, x_2, x_3) = 0$ otherwise. Summing $p_X(x_1, x_2, x_3)$ over all (x_1, x_2, x_3) satisfying $x_1 \neq x_2$ and $x_2 \neq x_3$ (it's most convenient to sum over x_2 in the outer sum), we have

$$
\begin{aligned}
P(X_1 \neq X_2, X_2 \neq X_3) &= \sum_{x_2=0}^{\infty} \sum_{\substack{x_1=0 \\ x_1 \neq x_2}}^{\infty} \sum_{\substack{x_3=0 \\ x_3 \neq x_2}}^{\infty} \left(\frac{1}{2}\right)^{x_1+1} \left(\frac{1}{2}\right)^{x_2+1} \left(\frac{1}{2}\right)^{x_3+1} \\
&= \sum_{x_2=0}^{\infty} \sum_{\substack{x_1=0 \\ x_1 \neq x_2}}^{\infty} \left(\frac{1}{2}\right)^{x_1+1} \left(\frac{1}{2}\right)^{x_2+1} \left(1 - \left(\frac{1}{2}\right)^{x_2+1}\right) \\
&= \sum_{x_2=0}^{\infty} \left(\frac{1}{2}\right)^{x_2+1} \left(1 - \left(\frac{1}{2}\right)^{x_2+1}\right)^2 \\
&= \sum_{x_2=0}^{\infty} \left[\left(\frac{1}{2}\right)^{x_2+1} - 2\left(\frac{1}{2}\right)^{2x_2+2} + \left(\frac{1}{2}\right)^{3x_2+3}\right] \\
&= 1 - 2\left(\frac{1}{2}\right)^2 \frac{1}{1-1/4} + \left(\frac{1}{2}\right)^3 \frac{1}{1-1/8} = 1 - \frac{2}{3} + \frac{1}{7} = \frac{10}{21}.
\end{aligned}
$$

(b) We want to compute $E[(X_1 - X_2)^2 + (X_2 - X_3)^2] = E[(X_1 - X_2)^2] + E[(X_2 - X_3)^2] = 2E[(X_1 - X_2)^2]$, where the last equality follows because (X_1, X_2) and (X_2, X_3) are identically distributed. So we can compute the answer as

$$
\begin{aligned}
2E[(X_1 - X_2)^2] &= 2\big(E[X_1^2] - 2E[X_1 X_2] + E[X_2^2]\big) \\
&= 2\big(2E[X_1^2] - 2E[X_1]E[X_2]\big) \quad \text{(since } X_1, X_2 \text{ are i.i.d.)} \\
&= 4\mathrm{Var}(X_1) \quad \text{(since } E[X_1] = E[X_2]) \\
&= 4 \times 2 = 8.
\end{aligned}
$$

2. Let $X_1, X_2.X_3$ be independent and identically distributed continuous random variables with common pdf $f_X(x) = \frac{1}{x^2}$ for $x \geq 1$ and $f_X(x) = 0$ for $x < 1$. Find $P(X_1X_2X_3 \leq e)$, where $e = \exp(1)$. (I want you to write out the appropriate 3-dimensional integral and evaluate it. Here's a fact you may find useful: $\frac{d}{dx}\frac{(\ln x)^2}{2} = \frac{\ln x}{x}$). [10]

Solution: Since X_1, X_2, X_3 are independent with common marginal pdf $f_X(x)$, the joint pdf of $(X_1, X_2, X_3)^T$ is $f_X(x_1)f_X(x_2)f_X(x_3) = \frac{1}{x_1^2x_2^2x_3^2}$ for $x_1, x_2, x_3 \geq 1$, and is equal to 0 otherwise. $P(X_1X_2X_3 \leq e)$ is found by integrating the joint pdf over all $(x_1, x_2, x_3)^T \in \mathbb{R}^3$ that satisfy $x_1x_2x_3 \leq e$. This triple integral is evaluated as

$$
\begin{aligned}
P(X_1X_2X_3 \leq e) &= \int_1^e \int_1^{e/x_3} \int_1^{e/(x_2x_3)} \frac{1}{x_1^2}\frac{1}{x_2^2}\frac{1}{x_3^2} dx_1 dx_2 dx_3 \\
&= \int_1^e \int_1^{e/x_3} \left[-\frac{1}{x_1}\bigg|_1^{e/(x_2x_3)} \right] \frac{1}{x_2^2}\frac{1}{x_3^2} dx_2 dx_3 \\
&= \int_1^e \int_1^{e/x_3} \left(1 - \frac{x_2x_3}{e}\right) \frac{1}{x_2^2}\frac{1}{x_3^2} dx_2 dx_3 \\
&= \int_1^e \left[-\frac{1}{x_2}\bigg|_1^{e/x_3} \right] \frac{1}{x_3^2} dx_3 - \frac{1}{e}\int_1^e \left[\ln x_2 \bigg|_1^{e/x_3} \right] \frac{1}{x_3} dx_3 \\
&= \int_1^e \left(1 - \frac{x_3}{e}\right) \frac{1}{x_3^2} dx_3 - \frac{1}{e}\int_1^e \ln\left(\frac{e}{x_3}\right) \frac{1}{x_3} dx_3 \\
&= 1 - \frac{1}{e} - \frac{1}{e}\ln e - \frac{1}{e}\left[(\ln e)(\ln e) - \int_1^e \frac{\ln x_3}{x_3} dx_3 \right] \\
&= 1 - \frac{1}{e} - \frac{1}{e} - \frac{1}{e} + \frac{1}{e}\frac{(\ln e)^2}{2} = 1 - \frac{2.5}{e} \approx .0803.
\end{aligned}
$$

3. The times to failure of components are independent exponentially distributed random variables with parameter λ (the Exponential(λ) distribution has pdf $f(t) = \lambda e^{-\lambda t}$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$; df $F(t) = 1 - e^{-\lambda t}$ for $t > 0$ and $F(t) = 0$ for $t \leq 0$; and mean $\frac{1}{\lambda}$). Suppose system A consists of two components and fails as soon as one component fails, while system B consists of two components and fails as soon as both components fail. Suppose both systems start with brand new components. Let X denote the lifetime of system A and Y denote the lifetime of system B.

(a) Find $E[X - Y]$. [5]

(b) Find the probability that system B lasts at least twice as long as system A. [5]

Solution:

(a) We wish to find $E[X]$ and $E[Y]$. Let X_1, X_2 denote the lifetimes of the components in system A and let Y_1, Y_2 denote the lifetimes of the components in system B. Then $X = \min(X_1, X_2)$ and $Y = \max(Y_1, Y_2)$. Since X_1, X_2, Y_1, Y_2 are independent exponentially distributed random variables with parameter λ, we have that X has pdf $f_X(x) = 2f(x)(1 - F(x)) = 2\lambda e^{-\lambda x}e^{-\lambda x} = 2\lambda e^{-2\lambda x}$ for $x > 0$ and $f_X(x) = 0$ for $x \leq 0$. The pdf of Y is $f_Y(y) = 2f(y)F(y) = 2\lambda e^{-\lambda y}(1 - e^{-\lambda y})$ for $y > 0$ and $f_Y(y) = 0$ for $y \leq 0$. Note that $X \sim$ Exponential(2λ) so $E[X] = \frac{1}{2\lambda}$. The expected value of Y is

$$E[Y] = \int_0^\infty y 2\lambda e^{-\lambda y}(1 - e^{-\lambda y})dy = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}.$$

Then $E[X - Y] = E[X] - E[Y] = \frac{1}{2\lambda} - \frac{3}{2\lambda} = -\frac{1}{\lambda}$.

(b) Note that X and Y are independent, so their joint pdf is the product of their marginal pdfs. We want $P(Y \geq 2X)$. This is the two-dimensional integral

$$\begin{aligned}
P(Y \geq 2X) &= \int_0^\infty \int_{2x}^\infty f_X(x)f_Y(y)dydx = \int_0^\infty f_X(x) \int_{2x}^\infty f_Y(y)dydx \\
&= \int_0^\infty f_X(x)(1 - F_Y(2x))dx = \int_0^\infty 2\lambda e^{-2\lambda x}(1 - (1 - e^{-2\lambda x})^2)dx \\
&= \int_0^\infty 2\lambda e^{-2\lambda x}(2e^{-2\lambda x} - e^{-4\lambda x})dx = \int_0^\infty 4\lambda e^{-4\lambda x}dx - \frac{1}{3}\int_0^\infty 6\lambda e^{-6\lambda x}dx \\
&= 1 - \frac{1}{3} = \frac{2}{3}.
\end{aligned}$$

where $F_Y(\cdot)$ is the df of Y, given by

$$F_Y(y) = P(Y \leq y) = \begin{cases} (1 - e^{-\lambda y})^2 & \text{for } y > 0 \\ 0 & \text{for } y \leq 0. \end{cases}$$