Queen's University

Department of Mathematics and Statistics

MTHE/STAT 353

Midterm Examination Solutions 2021

- **THE SOLUTIONS ARE TO BE UPLOADED TO CROWDMARK**. You should have received an invitation from Crowdmark to submit your solutions.
- **Duration = 120 minutes + 30 minutes to prepare your solutions for upload to Crowdmark and to submit your solutions to Crowdmark.**
- Write your solutions using your own paper. A tablet such as an ipad is **not allowed**. Use a dark lead pencil or pen to write your solutions. Solutions which cannot be read because they are too faint will not be marked.
- Write your student number at the start of each solution and number each solution.
- Total points = 30. Each question is worth 10 marks. Marks per part question are shown in brackets at the right.
- This is a closed book exam.
- One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
- A simple calculator is permitted.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.

1. Let $Y_1, \ldots, Y_n$ ($n \geq 2$) be independent and identically distributed discrete random variables with common pmf $f_Y(y) = \left(\frac{1}{2}\right)^{y+1}$ for $y = 0, 1, 2, \ldots$ and $f_Y(y) = 0$ otherwise (the mean of this distribution is 1). Let $X = \sum_{k=1}^{n-1} |Y_{k+1} - Y_k|$ ($|\cdot|$ is absolute value).

(a) Find $P(X = 0)$. [5]

(b) Find $E[X]$. [5]

Solution:

(a) The event $\{X = 0\}$ is the same as the event $\{Y_1 = \ldots = Y_n\}$. Since $Y_1, \ldots, Y_n$ are independent their joint pmf is the product of their marginal pmfs. Thus, we have

$$\begin{aligned}
P(X = 0) &= P(Y_1 = \ldots = Y_n) \\
&= \sum_{y=0}^{\infty} P(Y_1 = y, \ldots, Y_n = y) \\
&= \sum_{y=0}^{\infty} P(Y_1 = y)^n \quad \text{(since the } Y_i\text{s are i.i.d.)} \\
&= \sum_{y=0}^{\infty} \left(\left(\frac{1}{2}\right)^{y+1}\right)^n \\
&= \left(\frac{1}{2}\right)^n \sum_{y=0}^{\infty} \left(\left(\frac{1}{2}\right)^n\right)^y = \left(\frac{1}{2}\right)^n \frac{1}{1 - (1/2)^n} = \frac{1}{2^n - 1}.
\end{aligned}$$

(b) By linearity of expectation $E[X] = \sum_{k=1}^{n-1} E[|Y_{k+1} - Y_k|]$. To compute $E[|Y_{k+1} - Y_k|]$ we divide the set of possible (y_k, y_{k+1}) pairs into those for which $y_{k+1} > y_k$ and those for which $y_{k+1} < y_k$ (those for which $y_{k+1} = y_k$ will not enter into the expectation as the absolute difference will contribute 0). Also, by symmetry, the sums over these 2 sets will be the same. We have

$$\begin{aligned}
E[|Y_{k+1} - Y_k|] &= 2 \sum_{y_k=0}^{\infty} \sum_{y_{k+1}=y_k+1}^{\infty} (y_{k+1} - y_k) \left(\frac{1}{2}\right)^{y_k+1} \left(\frac{1}{2}\right)^{y_{k+1}+1} \\
&= 2 \sum_{y_k=0}^{\infty} \left(\frac{1}{2}\right)^{y_k+1} \sum_{j=1}^{\infty} j \left(\frac{1}{2}\right)^{y_k+1+j} \qquad (j = y_{k+1} - y_k) \\
&= 2 \sum_{y_k=0}^{\infty} \left(\frac{1}{2}\right)^{2y_k+1} \qquad \text{(the remaining sum is equal to 1)} \\
&= \sum_{y_k=0}^{\infty} \left(\frac{1}{4}\right)^{y_k} = \frac{1}{1 - 1/4} = \frac{4}{3}.
\end{aligned}$$

Since this does not depend on k we have $E[X] = \frac{4(n-1)}{3}$.

2. Let $X_1, X_2.X_3$ be independent and identically distributed Exponential(λ) random variables (the Exponential(λ) distribution has pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f_X(x) = 0$ for $x \leq 0$, and df $F_X(x) = 1 - e^{-\lambda x}$ for $x > 0$ and $F_X(x) = 0$ for $x \leq 0$). Find $P(X_1 + X_2 + X_3 \leq \frac{3}{2})$. (I want you to write out the appropriate 3-dimensional integral and evaluate it). [10]

Solution: The appropriate 3-dimensional integral is (by symmetry every order in which the integrations are done is equally easy)

$$
\begin{aligned}
& P\left(X_1 + X_2 + X_3 \leq \frac{3}{2}\right) \\
= \ & \int_0^{3/2} \int_0^{3/2 - x_3} \int_0^{3/2 - x_2 - x_3} f_X(x_1) f_X(x_2) f_X(x_3) dx_1 dx_2 dx_3 \\
= \ & \int_0^{3/2} \int_0^{3/2 - x_3} f_X(x_2) f_X(x_3)(1 - e^{-\lambda(3/2 - x_2 - x_3)}) dx_2 dx_3 \\
= \ & \int_0^{3/2} \lambda e^{-\lambda x_3} \int_0^{3/2 - x_3} \left[\lambda e^{-\lambda x_2} - \lambda e^{-\lambda(3/2 - x_3)}\right] dx_2 dx_3 \\
= \ & \int_0^{3/2} \lambda e^{-\lambda x_3} \left[1 - e^{-\lambda(3/2 - x_3)} - \lambda\left(\frac{3}{2} - x_3\right) e^{-\lambda(3/2 - x_3)}\right] dx_3 \\
= \ & 1 - e^{-3\lambda/2} - \frac{3\lambda}{2} e^{-3\lambda/2} - \left(\frac{3\lambda}{2}\right)^2 e^{-3\lambda/2} + \lambda^2 e^{-3\lambda/2} \frac{(3/2)^2}{2} \\
= \ & 1 - e^{-3\lambda/2}\left[1 + \frac{3\lambda}{2} + \frac{1}{2}\left(\frac{3\lambda}{2}\right)^2\right].
\end{aligned}
$$

3. Let $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n$ be independent and identically distributed continuous random variables with common pdf f, where $f(t) = \frac{1}{t^2} e^{-1/t}$ for $t > 0$ and $f(t) = 0$ for $t \le 0$. Let $X = \min\big\{\max(Y_1, Z_1), \ldots, \max(Y_n, Z_n)\big\}$.

(a) Find the pdf of X. [5]

(b) The median of the distribution of X is that value m satisfying $P(X \le m) = \frac{1}{2}$. Find the median when $n = 5$. [5]

Solution:

(a) First, the common df of the Y_i's and Z_i's is $F(t) = e^{-1/t}$ for $t > 0$ and $F(t) = 0$ for $t \le 0$. Let $W_i = \max(Y_i, Z_i)$, for $i = 1, \ldots, n$. Then W_i has df $F_W(t) = F(t)^2 = e^{-2/t}$ for $t > 0$ and $F_W(t) = 0$ for $t \le 0$. The pdf of W_i is then $f_W(t) = \frac{2}{t^2} e^{-2/t}$ for $t > 0$ and $f_W(t) = 0$ for $t \le 0$. Next, note that $W_1, \ldots, W_n$ are mutually independent. Then the pdf of $X = \min(W_1, \ldots, W_n)$ is

$$f_X(x) = n f_W(x)(1 - F_W(x))^{n-1} = \frac{2n}{x^2} e^{-2/x}(1 - e^{-2/x})^{n-1}$$

for $x > 0$ and $f_X(x) = 0$ for $x \le 0$.

(b) The df of X is $F_X(x) = 1 - (1 - F_W(x))^n = 1 - (1 - e^{-2/x})^n$ for $x > 0$ and $F_X(x) = 0$ for $x \le 0$. The median m of the distribution of X when $n = 5$ satisfies

$$\begin{aligned}
1 - (1 - e^{-2/m})^5 = \frac{1}{2} \quad &\Leftrightarrow \quad \left(\frac{1}{2}\right)^{1/5} = 1 - e^{-2/m} \\
&\Leftrightarrow \quad e^{-2/m} = 1 - \left(\frac{1}{2}\right)^{1/5} \\
&\Leftrightarrow \quad \frac{2}{m} = -\ln\left(1 - \left(\frac{1}{2}\right)^{1/5}\right) \\
&\Leftrightarrow \quad m = \frac{2}{-\ln(1 - .5^{1/5})} \approx .978
\end{aligned}$$