Practice Problems for Final Exam

Broad list of topics covered by the final exam:

(A) Filters to eliminate or pass trends and/or seasonal components

(B) ACVF and ACF of stationary process

(C) AR(1), MA(q), ARMA(1,1) processes.

(D) Distribution and properties of sample mean and sample autocorrelations.

(E) Linear prediction.

Problems may require the use of R.

Will not cover
- innovations algorithm
- tests from Section 1.6

① Topic A

(a) (i) Show that a causal filter $\{a_j\}_{j=0}^{\infty}$ will pass an arbitrary quadratic polynomial, say $m_t = c_0 + c_1 t + c_2 t^2$, if the following 3 conditions are satisfied: $\sum_{j=0}^{\infty} a_j = 1$, $\sum_{j=0}^{\infty} j a_j = 0$, $\sum_{j=0}^{\infty} j^2 a_j = 0$.

Sol. Let $a_j = 0$ for $j<0$, then assuming the conditions are satisfied, $\sum_{j=-\infty}^{\infty} a_j = 1$, $\sum_{j=-\infty}^{\infty} j a_j = 0$, $\sum_{j=-\infty}^{\infty} j^2 a_j = 0$ will hold.

By Problem 1.12, the linear filter $\{a_j\}_{j=-\infty}^{\infty}$ will pass arbitrary quadratic polynomials.

Second method: Directly apply the filter to m_t. The output is

$$\sum_{j=0}^{\infty} a_j m_{t-j} = \sum_{j=0}^{\infty} a_j (c_0 + c_1(t-j) + c_2(t-j)^2)$$

$$= (c_0 + c_1 t + c_2 t^2) \underbrace{\sum_{j=0}^{\infty} a_j}_{1} - (c_1 + 2c_2 t) \underbrace{\sum_{j=0}^{\infty} j a_j}_{0} + c_2 \underbrace{\sum_{j=0}^{\infty} j^2 a_j}_{0}$$

$$= c_0 + c_1 t + c_2 t^2$$

$$= m_t$$

Thus, the causal filter $\{a_j\}_{j=0}^{\infty}$ passes arbitrary quadratic polynomials.

(ii) Find a causal filter $\{a_0, a_1, \ldots, a_6\}$ with $a_j \neq 0$ for $j = 0, \ldots, 6$ (and $a_j = 0$ for $j \notin \{0, 1, \ldots, 6\}$) that passes arbitrary quadratic polynomials and eliminates seasonal components with period 5.

Sol. First, apply the filter $1 + B + B^2 + B^3 + B^4$, which will eliminate the seasonal component with period 5. Then apply a length 3 filter $\{b_0, b_1, b_2\}$ to get the composite filter

$$(b_0 + b_1 B + b_2 B^2)(1 + B + B^2 + B^3 + B^4) = \sum_{j=0}^{6} a_j B^j$$

where

$$\begin{aligned}
a_0 &= b_0 \\
a_1 &= b_0 + b_1 \\
a_2 &= b_0 + b_1 + b_2 \\
a_3 &= b_0 + b_1 + b_2 \\
a_4 &= b_0 + b_1 + b_2 \\
a_5 &= b_1 + b_2 \\
a_6 &= b_2
\end{aligned}$$

By part (i), the filter $\{a_0, a_1, \ldots, a_6\}$ will pass arbitrary quadratic polynomials if

$$\begin{aligned}
5(b_0 + b_1 + b_2) &= 1 \\
10 b_0 + 15 b_1 + 20 b_2 &= 0 \\
30 b_0 + 55 b_1 + 90 b_2 &= 0
\end{aligned}$$

These equations have solution $(b_0, b_1, b_2)^T = (1, -1.2, .4)^T$. Then the filter $\{a_0, a_1, \ldots, a_6\}$ is given by

$$\{a_0, a_1, \ldots, a_6\} = \{1, -.2, .2, .2, .2, -.8, .4\}.$$

(Remark: One might try to eliminate the seasonal component first using the lag 5 difference operator $\nabla_5 = 1 - B^5$. However, any causal filter we then apply to the output can only have 2 coefficients b_0 and b_1 (otherwise the filter would have powers of B greater than 6). You will find that with only b_0 and b_1, you will not be able to come up with an overall filter $\{a_0, a_1, \ldots, a_6\}$ that satisfy the sufficient conditions from part (i).

(b) Find a symmetric filter $\{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3\}$ where $a_j \neq 0$ for $j=-3,\dots,3$ and $a_{-j}=a_j$ that passes arbitrary cubic polynomials and eliminates seasonal components with period either 2 or 3.

Sol. By Problem 1.12, the desired symmetric filter should have coefficients a_0, a_1, a_2, a_3 that satisfy

① $\quad a_0 + 2a_1 + 2a_2 + 2a_3 = 1$

② $\quad a_1 + 4a_2 + 9a_3 = 0$

When applied to a seasonal component $\{s_t\}$ the output will be

$$a_3 s_{t-3} + a_2 s_{t-2} + a_1 s_{t-1} + a_0 s_t + a_1 s_{t+1} + a_2 s_{t+2} + a_3 s_{t+3}$$

$$= \begin{cases} (a_0 + 2a_3)s_t + (a_1+a_2)s_{t-1} + (a_1+a_2)s_{t-2} & \text{if period is 3} \\ (a_0 + 2a_2)s_t + 2(a_1+a_3)s_{t-1} & \text{if period is 2} \end{cases}$$

From the above, to eliminate a seasonal component with either period 2 or 3, it is sufficient to impose the further constraints

③ $\quad a_0 + 2a_3 = a_1 + a_2$

④ $\quad a_0 + 2a_2 = 2(a_1 + a_3)$

The solution to the system of 4 linear equations given by ①–④ is

$$(.43055,\ .29861,\ .03472,\ -.04861)^T = (a_0, a_1, a_2, a_3)^T$$

Then, the desired symmetric filter is given by

$$\{\underset{a_{-3}}{-.04861},\ \underset{a_{-2}}{.03472},\ \underset{a_{-1}}{.29861},\ \underset{a_0}{.43055},\ \underset{a_1}{.29861},\ \underset{a_2}{.03472},\ \underset{a_3}{-.04861}\}.$$

Note. Another component to this question could be to code the filter you come up with in R and demonstrate that it has the required properties.

② Topics B, C

(a) Let $\{X_t\}$ be an MA(2) process with parameters θ_1, θ_2 and σ^2,

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\}$ is a zero-mean $WN(\sigma^2)$.

Find $\gamma_X(h)$ and $\rho_X(h)$, the ACVF and ACF, respectively, of $\{X_t\}$ for $h = 0, \pm 1, \pm 2, \ldots$

Sol. We have

$$Var(X_t) = \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 = \sigma^2(1 + \theta_1^2 + \theta_2^2).$$

$$Cov(X_t, X_{t-1}) = Cov(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-1} + \theta_1 Z_{t-2} + \theta_2 Z_{t-3})$$

$$= Cov(\theta_1 Z_{t-1}, Z_{t-1}) + Cov(\theta_2 Z_{t-2}, \theta_1 Z_{t-2})$$

$$= \theta_1 \sigma^2 + \theta_1 \theta_2 \sigma^2 = \sigma^2 \theta_1 (1 + \theta_2)$$

$$Cov(X_t, X_{t-2}) = Cov(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-2} + \theta_1 Z_{t-3} + \theta_2 Z_{t-4})$$

$$= Cov(\theta_2 Z_{t-2}, Z_{t-2})$$

$$= \theta_2 \sigma^2$$

Then
$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta_1^2 + \theta_2^2) & h = 0 \\ \sigma^2 \theta_1 (1 + \theta_2) & h = \pm 1 \\ \sigma^2 \theta_2 & h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

and
$$\rho_X(h) = \begin{cases} 1 & h = 0 \\ \dfrac{\theta_1(1 + \theta_2)}{1 + \theta_1^2 + \theta_2^2} & h = \pm 1 \\ \dfrac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Let $\{Z_t\}$ be a zero mean $WN(\sigma^2)$ process, and let $\{Y_t\}$ be the MA(1) process $Y_t = Z_t + \frac{1}{2} Z_{t-1}$. Let $X_t = Y_t + a Y_{t-1}$, where a is a constant. Show that $\{X_t\}$ is an MA(2) process and find the parameters θ_1 and θ_2 of this process in terms of a. Find a so that $\{X_t\}$ has ACF with $\rho_X(0) = 1$, $\rho_X(1) = -\frac{1}{6}$ and $\rho_X(2) = -\frac{1}{3}$.

Sol. We have

$$X_t = Y_t + a Y_{t-1} = Z_t + \tfrac{1}{2} Z_{t-1} + a\left(Z_{t-1} + \tfrac{1}{2} Z_{t-2}\right)$$

$$= Z_t + \left(a + \tfrac{1}{2}\right) Z_{t-1} + \frac{a}{2} Z_{t-2}.$$

Thus, $\{X_t\}$ is an MA(2) process with parameter $\theta_1 = a + \frac{1}{2}$ and $\theta_2 = \frac{a}{2}$.

From part (a), if we want $\rho_X(1) = -\frac{1}{6}$ and $\rho_X(2) = -\frac{1}{3}$, then we need

$$(1) \quad \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} = -\frac{1}{6} \quad \text{and} \quad (2) \quad \frac{\theta_2}{1+\theta_1^2+\theta_2^2} = -\frac{1}{3}$$

In terms of a, (1) is

$$\frac{\left(a+\frac{1}{2}\right)\left(1+\frac{a}{2}\right)}{1+\left(a+\frac{1}{2}\right)^2+\frac{a^2}{4}} = -\frac{1}{6}$$

or

$$\frac{a + \frac{a^2}{2} + \frac{1}{2} + \frac{a}{4}}{1 + a^2 + a + \frac{1}{4} + \frac{a^2}{4}} = -\frac{1}{6}$$

or

$$\frac{5a + 2a^2 + 2}{5 + 5a^2 + 4a} = -\frac{1}{6} \qquad (3)$$

and (2) is

$$\frac{\frac{a}{2}}{1 + a^2 + a + \frac{1}{4} + \frac{a^2}{4}} = -\frac{1}{3}$$

or

$$\frac{2a}{5 + 5a^2 + 4a} = -\frac{1}{3} \qquad (4)$$

Using Eq. (4), we have $-6a = 5 + 5a^2 + 4a$ or $5a^2 + 10a + 5 = 0$

or $a^2 + 2a + 1 = 0$

or $(a+1)^2 = 0$

$\Rightarrow a = -1$

Check that $a = -1$ satisfies (3):

$$\frac{-5+2+2}{5+5-4} = -\frac{1}{6} \quad \text{or} \quad \frac{-1}{6} = -\frac{1}{6} \quad \checkmark$$

So $a = -1$ satisfies both (3) and (4). So we can conclude that $\{X_t\}$ with $a = -1$ has the given ACF.