Final Exam Practice Problems (Part 2)

(3) Topics B, C, D

Suppose $\{X_t\}$ is an MA(2) process with parameters θ_1, θ_2 and σ^2, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\}$ is a zero-mean $WN(\sigma^2)$.

(a) Let $\rho_X(h)$ be the ACF of $\{X_t\}$. Show that $|\rho_X(1)| \le \frac{1}{\sqrt{2}}$ and $|\rho_X(2)| \le \frac{1}{2}$ for any θ_1, θ_2, and σ^2.

Sol. First, we have that $\rho_X(1) = \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$ and $\rho_X(2) = \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2}$.

First, consider $\rho_X(2)$. We first note that we should have $\theta_2 > 0$ to maximize $\rho_X(2)$. Further, for any $\theta_2 > 0$, $\rho_X(2)$ will be maximized over θ_1 by taking $\theta_1 = 0$. Therefore, the maximum of $\rho_X(2)$ is $\max\limits_{\theta_2>0} \dfrac{\theta_2}{1+\theta_2^2}$. Taking the derivative w.r.t. θ_2 and setting this to 0 gives

$$\frac{(1+\theta_2^2) - \theta_2 2\theta_2}{(1+\theta_2^2)^2} = 0 \iff 1+\theta_2^2 - 2\theta_2^2 = 0 \iff 1 = \theta_2^2$$

So $\theta_2 = +1$ (and $\theta_1 = 0$) maximizes $\rho_X(2)$. The maximum value is $\dfrac{1}{1+0^2+1^2} = \dfrac{1}{2}$. Therefore $|\rho_X(2)| \le \frac{1}{2}$.

Now, consider $\rho_X(1)$. First, let us note that to maximize $\rho_X(1)$ over θ_1 and θ_2 we can assume that $\theta_2 \ne -1$ because when $\theta_2 = -1$, $\rho_X(1) = 0$ (which is clearly not the maximum).

For a fixed $\theta_2 \ne -1$, differentiating $\rho_X(1)$ w.r.t. θ_1 and setting to 0 gives

$$\frac{(1+\theta_1^2+\theta_2^2)(1+\theta_2) - \theta_1(1+\theta_2)2\theta_1}{(1+\theta_1^2+\theta_2^2)^2} = 0$$

$$\iff 1+\theta_1^2+\theta_2^2 = 2\theta_1^2 \quad \text{since } 1+\theta_2 \ne 0$$

$$\iff 1+\theta_2^2 = \theta_1^2$$

Thus, we get that $\theta_1 = \pm\sqrt{1+\theta_2^2}$ will potentially maximize $\rho_X(1)$ for a given value of θ_2. Let $\rho_X(1|\theta_2)$ be the maximum value of $\rho_X(1)$ for the given θ_2. Thus, we wish to maximize $\rho_X(1|\theta_2)$ over θ_2.

For a fixed θ_2, when we plug in the values $\theta_1 = \pm\sqrt{1+\theta_2^2}$ we get $\dfrac{\pm\sqrt{1+\theta_2^2}\,(1+\theta_2)}{1+(1+\theta_2^2)+\theta_2^2}$. We wish to maximize this over $\theta_2 \neq -1$. Rather than maximizing the above over θ_2, we will maximize the square of the above over θ_2, since the square is a monotone function so the value of θ_2 that maximizes the square also maximizes the above. Thus, we wish to maximize

$$(*)\quad \frac{(1+\theta_2^2)(1+\theta_2)^2}{(2(1+\theta_2^2))^2} = \frac{(1+\theta_2)^2}{4(1+\theta_2^2)} = \frac{(1+\theta_2^2)+2\theta_2}{4(1+\theta_2^2)} = \frac{1}{4} + \frac{\theta_2}{2(1+\theta_2^2)}$$

We have already computed that $\dfrac{\theta_2}{1+\theta_2^2}$ is maximized at $\theta_2 = 1$, and the maximum value of $\dfrac{\theta_2}{1+\theta_2^2}$ is $\frac{1}{2}$. Therefore, $(*)$ is maximized at $\theta_2 = 1$, and its maximum value is $\frac{1}{4} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$.
Since $(*)$ is the square of $\rho_X(1)$ at a given θ_2. Therefore, the maximum value of $\rho_X(1)^2$ is $\frac{1}{2}$, which gives that the maximum value of $|\rho_X(1)|$ is $\frac{1}{\sqrt{2}}$. Therefore, $|\rho_X(1)| \le \frac{1}{\sqrt{2}}$.

(b) Let $\{X_t\}$ be an MA(2) process with MA parameters $\theta_1 = -\frac{1}{2}$ and $\theta_2 = -\frac{1}{2}$. Let $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ denote the sample autocorrelations based on $X_1, \ldots, X_n$. Find the approximate variances of $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ and construct 95% confidence intervals for $\rho_X(1)$ and $\rho_X(2)$ based on these.

Sol. From Section 2.4.2, the approximate variance of $\hat{\rho}_X(i)$ is $\frac{1}{n} w_{ii}$, where

$$w_{ii} = \sum_{k=1}^{\infty} \left(\rho_X(k+i) + \rho_X(k-i) - 2\rho_X(i)\rho_X(k)\right)^2$$

We wish to compute w_{11} and w_{22}.

For the MA(2) process with parameters θ_1 and θ_2

$$\rho_x(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \quad \text{and} \quad \rho_x(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$$

with $\theta_1 = -\frac{1}{2}$ and $\theta_2 = -\frac{1}{2}$, we have

$$\rho_x(1) = \frac{-\frac{1}{2}\left(\frac{1}{2}\right)}{1+\frac{1}{4}+\frac{1}{4}} = \frac{-\frac{1}{4}}{\frac{6}{4}} = -\frac{1}{6}, \quad \rho_x(2) = \frac{-\frac{1}{2}}{1+\frac{1}{4}+\frac{1}{4}} = \frac{-\frac{2}{4}}{\frac{6}{4}} = -\frac{1}{3}$$

So, to evaluate $W_{ii} = \sum_{k=1}^{\infty}\left(\rho_x(k+i) + \rho_x(k-i) - 2\rho_x(i)\rho_x(k)\right)^2$, we have

for $i=1$:

$$W_{11} = \left(\rho_x(2) + \rho_x(0) - 2\rho_x(1)^2\right)^2 + \left(\rho_x(1) - 2\rho_x(1)\rho_x(2)\right)^2 + \rho_x(2)^2$$

for $i=2$

$$W_{22} = \left(\rho_x(1) - 2\rho_x(2)\rho_x(1)\right)^2 + \left(\rho_x(0) - 2\rho_x(2)^2\right)^2 + \rho_x(1)^2 + \rho_x(2)^2$$

Plugging in $\rho_x(1) = -\frac{1}{6}$ and $\rho_x(2) = -\frac{1}{3}$ we have

$$W_{11} = \left(-\frac{1}{3} + 1 - 2\frac{1}{36}\right)^2 + \left(-\frac{1}{6} - 2\frac{1}{18}\right)^2 + \frac{1}{9}$$

$$= \left(\frac{11}{18}\right)^2 + \left(\frac{5}{18}\right)^2 + \frac{1}{9} = \frac{121+25+36}{324} = \frac{182}{324} = \frac{91}{162} \simeq .5617$$

$$W_{22} = \left(-\frac{1}{6} - 2\frac{1}{18}\right)^2 + \left(1 - 2\frac{1}{9}\right)^2 + \frac{1}{36} + \frac{1}{9}$$

$$= \left(\frac{5}{18}\right)^2 + \left(\frac{7}{9}\right)^2 + \frac{1}{36} + \frac{1}{9} = \frac{25+196+9+36}{324} = \frac{266}{324} = \frac{133}{162} \simeq .906$$

Therefore, $Var(\hat{\rho}_x(1)) \simeq \frac{.5617}{n}$ and $Var(\hat{\rho}_x(2)) \simeq \frac{.906}{n}$

From Section 2.4.2, we have that

$$\hat{\rho}_x(1) \dot\sim N\left(\rho_x(1), \frac{.5617}{n}\right) \quad \text{and} \quad \hat{\rho}_x(2) \dot\sim N\left(\rho_x(2), \frac{.906}{n}\right)$$

So

$$P\left(-1.96 < \frac{\hat{\rho}_x(1) - \rho_x(1)}{\sqrt{.5617/n}} < 1.96\right) \simeq .95$$

and

$$P\left(-1.96 < \frac{\hat{\rho}_x(2) - \rho_x(2)}{\sqrt{.906/n}} < 1.96\right) \simeq .95$$

The 95% confidence intervals based on these approximations are then

For $\rho_x(1)$: $\hat{\rho}_x(1) \pm 1.96\sqrt{.5617/n} = \hat{\rho}_x(1) \pm 1.469/\sqrt{n}$

For $\rho_x(2)$: $\hat{\rho}_x(2) \pm 1.96\sqrt{.906/n} = \hat{\rho}_x(2) \pm 1.776/\sqrt{n}$

(4) Let $\{X_t\}$ be an ARMA(1,1) process with parameters $\phi = \frac{1}{2}$ and $\theta = \frac{1}{2}$, and $\sigma^2 = 1$, i.e., $X_t - \frac{1}{2}X_{t-1} = Z_t + \frac{1}{2}Z_{t-1}$, where $\{Z_t\}$ is a zero-mean WN(1). The ACVF of the ARMA(1,1) process is

$$\gamma_X(0) = \sigma^2\left(1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right)$$

$$\gamma_X(1) = \sigma^2\left(\theta + \phi + \frac{(\theta+\phi)\phi}{1-\phi^2}\right)$$

$$\gamma_X(h) = \phi^{h-1}\gamma_X(1) \qquad h \geq 2$$

with $\phi = \frac{1}{2}$, $\theta = \frac{1}{2}$, $\sigma^2 = 1$

$$\gamma_X(0) = \frac{7}{3}$$

$$\gamma_X(1) = \frac{5}{3}$$

$$\gamma_X(h) = \left(\frac{1}{2}\right)^{h-1}\left(\frac{5}{3}\right)$$

(a) Give $P(X_2|X_1)$, the best linear predictor of X_2 in terms of X_1 (this is also denoted by P_1X_2). Compute the mean squared error of $P(X_2|X_1)$.

Sol. Recall, $P(X_2|X_1) = aX_1$, where a satisfies

$$\gamma_X(0)a = \gamma_X(1), \quad \text{or} \quad a = \frac{\gamma_X(1)}{\gamma_X(0)} = \frac{5/3}{7/3} = \frac{5}{7}.$$

So $P(X_2|X_1) = \frac{5}{7}X_1$. The minimum mean squared error is

$$E[(X_2 - P(X_2|X_1))^2] = \gamma_X(0) - \frac{5}{7}\gamma_X(1) = \frac{7}{3} - \left(\frac{5}{7}\right)\left(\frac{5}{3}\right) = \frac{24}{21} = \frac{8}{7}.$$

(b) Find $P(X_3|X_2,X_1)$. Compute the MSE of $P(X_3|X_2,X_1)$.

Sol. $P(X_3|X_2,X_1) = a_1X_2 + a_2X_1$, where $a = (a_1, a_2)^T$ satisfies

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} \iff \begin{bmatrix} 7/3 & 5/3 \\ 5/3 & 7/3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/6 \end{bmatrix}$$

$$\iff \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}^{-1}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 14 & -10 \\ -10 & 14 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 90 \\ -30 \end{bmatrix} = \begin{bmatrix} \frac{15}{16} \\ -\frac{5}{16} \end{bmatrix}.$$

Then $P(X_3|X_2,X_1) = \frac{15}{16}X_2 - \frac{5}{16}X_1$. The minimum MSE is

$$E[(X_3 - P(X_3|X_2,X_1))^2] = \frac{7}{3} - \left(\left(\frac{15}{16}\right)\left(\frac{5}{3}\right) - \left(\frac{5}{16}\right)\left(\frac{5}{6}\right)\right)$$

$$= \frac{224 - 150 + 25}{96} = \frac{99}{96} = \frac{33}{32} \approx 1.03125.$$

(c) Compute $P(X_n \mid X_2, X_1)$ and give the minimum MSE.

Compute the limit of this MSE as $n \to \infty$.

Sol. $P(X_n \mid X_2, X_1) = a_1 X_2 + a_2 X_1$ where $a = (a_1, a_2)^T$ satisfies

$$\begin{bmatrix} 7/3 & 5/3 \\ 5/3 & 7/3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(n-2) \\ \gamma_X(n-1) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^{n-3}\left(\frac{5}{3}\right) \\ \left(\frac{1}{2}\right)^{n-2}\left(\frac{5}{3}\right) \end{bmatrix}$$

or $$\begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Then $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \underbrace{\begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 5 \end{bmatrix}}_{\text{done in part (b)}} = \left(\frac{1}{2}\right)^{n-3} \begin{bmatrix} \frac{15}{16} \\ -\frac{5}{16} \end{bmatrix}$$

Then $P(X_n \mid X_2, X_1) = \left(\frac{1}{2}\right)^{n-3}\left(\frac{15}{16} X_2 - \frac{5}{16} X_1\right)$

The minimum MSE is

$$E[(X_n - P(X_n \mid X_2, X_1))^2] = \gamma_X(0) - \left(\frac{1}{2}\right)^{2n-6}\left(\left(\frac{15}{16}\right)\left(\frac{5}{3}\right) - \left(\frac{5}{16}\right)\left(\frac{5}{6}\right)\right)$$
$$= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \frac{150-25}{96}$$
$$= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \frac{125}{96}.$$

As $n \to \infty$, $E[(X_n - P(X_n \mid X_2, X_1))^2] \to \frac{7}{3}$.