Practice Problems for Final Exam

(1) (a) Find a causal filter $\{a_0, a_1, \dots, a_6\}$ with $a_j \neq 0$ for $j = 0, \dots, 6$ (and $a_j = 0$ for $j \notin \{0, 1, \dots, 6\}$) that passes arbitrary quadratic polynomials and eliminates seasonal components with period 5.

Sol. First, apply the filter $1 + B + B^2 + B^3 + B^4$, which will eliminate the seasonal component with period 5. Then apply a length 3 filter $\{b_0, b_1, b_2\}$ to get the composite filter

$$(b_0 + b_1 B + b_2 B^2)(1 + B + B^2 + B^3 + B^4) = \sum_{j=0}^{6} a_j B^j$$

where

$$\begin{aligned}
a_0 &= b_0 \\
a_1 &= b_0 + b_1 \\
a_2 &= b_0 + b_1 + b_2 \\
a_3 &= b_0 + b_1 + b_2 \\
a_4 &= b_0 + b_1 + b_2 \\
a_5 &= b_1 + b_2 \\
a_6 &= b_2
\end{aligned}$$

By part (i), the filter $\{a_0, a_1, \dots, a_6\}$ will pass arbitrary quadratic polynomials if

$$\begin{aligned}
5(b_0 + b_1 + b_2) &= 1 \\
10 b_0 + 15 b_1 + 20 b_2 &= 0 \\
30 b_0 + 55 b_1 + 90 b_2 &= 0
\end{aligned}$$

These equations have solution $(b_0, b_1, b_2)^T = (1, -1.2, .4)^T$. Then the filter $\{a_0, a_1, \dots, a_6\}$ is given by

$$\{a_0, a_1, \dots, a_6\} = \{1, -.2, .2, .2, .2, -.8, .4\}.$$

(Remark: One might try to eliminate the seasonal component first using the lag 5 difference operator $\nabla_5 = 1 - B^5$. However, any causal filter we then apply to the output can only have 2 coefficients b_0 and b_1 (otherwise the filter would have powers of B greater than 6). You will find that with only b_0 and b_1, you will not be able to come up with an overall filter $\{a_0, a_1, \dots, a_6\}$ that satisfy the requirements.)

(b) Find a symmetric filter $\{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3\}$ where $a_j \neq 0$ for $j = -3, \dots, 3$ and $a_{-j} = a_j$ that passes arbitrary cubic polynomials and eliminates seasonal components with period either 2 or 3.

Sol. By Problem 1.12, the desired symmetric filter should have coefficients a_0, a_1, a_2, a_3 that satisfy

① $\quad a_0 + 2a_1 + 2a_2 + 2a_3 = 1$

② $\quad a_1 + 4a_2 + 9a_3 = 0$

When applied to a seasonal component $\{s_t\}$ the output will be

$$a_3 s_{t-3} + a_2 s_{t-2} + a_1 s_{t-1} + a_0 s_t + a_1 s_{t+1} + a_2 s_{t+2} + a_3 s_{t+3}$$

$$= \begin{cases} (a_0 + 2a_3)s_t + (a_1 + a_2)s_{t-1} + (a_1 + a_2)s_{t-2} & \text{if period is 3} \\ (a_0 + 2a_2)s_t + 2(a_1 + a_3)s_{t-1} & \text{if period is 2} \end{cases}$$

From the above, to eliminate a seasonal component with either period 2 or 3, it is sufficient to impose the further constraints

③ $\quad a_0 + 2a_3 = a_1 + a_2$

④ $\quad a_0 + 2a_2 = 2(a_1 + a_3)$

The solution to the system of 4 linear equations given by ①–④ is

$$(.43055,\ .29861,\ .03472,\ -.04861)^T = (a_0, a_1, a_2, a_3)^T$$

Then, the desired symmetric filter is given by

$$\{\underset{a_{-3}}{-.04861},\ \underset{a_{-2}}{.03472},\ \underset{a_{-1}}{.29861},\ \underset{a_0}{.43055},\ \underset{a_1}{.29861},\ \underset{a_2}{.03472},\ \underset{a_3}{-.04861}\}$$

(2)

(a) Let $\{X_t\}$ be an MA(2) process with parameters θ_1, θ_2 and σ^2,
$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\}$ is a zero-mean $WN(\sigma^2)$.
Find $\gamma_X(h)$ and $\rho_X(h)$, the ACVF and ACF, respectively, of $\{X_t\}$ for $h = 0, \pm 1, \pm 2, \ldots$

Sol. We have

$$Var(X_t) = \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 = \sigma^2(1 + \theta_1^2 + \theta_2^2).$$

$$Cov(X_t, X_{t-1}) = Cov(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-1} + \theta_1 Z_{t-2} + \theta_2 Z_{t-3})$$

$$= Cov(\theta_1 Z_{t-1}, Z_{t-1}) + Cov(\theta_2 Z_{t-2}, \theta_1 Z_{t-2})$$

$$= \theta_1 \sigma^2 + \theta_1 \theta_2 \sigma^2 = \sigma^2 \theta_1 (1 + \theta_2)$$

$$Cov(X_t, X_{t-2}) = Cov(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-2} + \theta_1 Z_{t-3} + \theta_2 Z_{t-4})$$

$$= Cov(\theta_2 Z_{t-2}, Z_{t-2})$$

$$= \theta_2 \sigma^2$$

Then
$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta_1^2+\theta_2^2) & h = 0 \\ \sigma^2 \theta_1 (1+\theta_2) & h = \pm 1 \\ \sigma^2 \theta_2 & h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

and
$$\rho_X(h) = \begin{cases} 1 & h = 0 \\ \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} & h = \pm 1 \\ \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2} & h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Let $\{Z_t\}$ be a zero mean $WN(\sigma^2)$ process, and let $\{Y_t\}$ be the MA(1) process $Y_t = Z_t + \frac{1}{2} Z_{t-1}$. Let $X_t = Y_t + a Y_{t-1}$, where a is a constant. Show that $\{X_t\}$ is an MA(2) process and find the parameters θ_1 and θ_2 of this process in terms of a. Find a so that $\{X_t\}$ has ACF with $\rho_X(0) = 1$, $\rho_X(1) = -\frac{1}{6}$ and $\rho_X(2) = -\frac{1}{3}$.

Sol. We have

$$X_t = Y_t + a Y_{t-1} = Z_t + \frac{1}{2} Z_{t-1} + a\left(Z_{t-1} + \frac{1}{2} Z_{t-2}\right)$$

$$= Z_t + (a + \tfrac{1}{2}) Z_{t-1} + \frac{a}{2} Z_{t-2}.$$

Thus, $\{X_t\}$ is an MA(2) process with parameter $\theta_1 = a + \frac{1}{2}$ and $\theta_2 = \frac{a}{2}$.

From part (a), if we want $\rho_X(1) = -\frac{1}{6}$ and $\rho_X(2) = -\frac{1}{3}$, then we need

$$(1) \quad \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} = -\frac{1}{6} \quad \text{and} \quad (2) \quad \frac{\theta_2}{1+\theta_1^2+\theta_2^2} = -\frac{1}{3}$$

In terms of a, (1) is

$$\frac{(a+\frac{1}{2})(1+\frac{a}{2})}{1+(a+\frac{1}{2})^2+\frac{a^2}{4}} = -\frac{1}{6}$$

or

$$\frac{a + \frac{a^2}{2} + \frac{1}{2} + \frac{a}{4}}{1 + a^2 + a + \frac{1}{4} + \frac{a^2}{4}} = -\frac{1}{6}$$

or

$$\frac{5a + 2a^2 + 2}{5 + 5a^2 + 4a} = -\frac{1}{6} \qquad (3)$$

and (2) is

$$\frac{\frac{a}{2}}{1 + a^2 + a + \frac{1}{4} + \frac{a^2}{4}} = -\frac{1}{3}$$

or

$$\frac{2a}{5 + 5a^2 + 4a} = -\frac{1}{3} \qquad (4)$$

Using Eq. (4), we have $-6a = 5 + 5a^2 + 4a$ or $5a^2 + 10a + 5 = 0$

or $a^2 + 2a + 1 = 0$

or $(a+1)^2 = 0$

$\Rightarrow a = -1$

Check that $a = -1$ satisfies (3):

$$\frac{-5+2+2}{5+5-4} = -\frac{1}{6} \quad \text{or} \quad \frac{-1}{6} = -\frac{1}{6}$$

So $a = -1$ satisfies both (3) and (4). So we can conclude that $\{X_t\}$ with $a = -1$ has the given ACF.

③ Suppose $\{X_t\}$ is an MA(2) process with parameters θ_1, θ_2 and σ^2, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\}$ is a zero-mean $WN(\sigma^2)$.

(a) Let $\rho_X(h)$ be the ACF of $\{X_t\}$. Show that $|\rho_X(1)| \leq \frac{1}{\sqrt{2}}$ and $|\rho_X(2)| \leq \frac{1}{2}$ for any θ_1, θ_2, and σ^2.

Sol. First, we have that $\rho_X(1) = \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$ and $\rho_X(2) = \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2}$.

First, consider $\rho_X(2)$. We first note that we should have $\theta_2 > 0$ to maximize $\rho_X(2)$. Further, for any $\theta_2 > 0$, $\rho_X(2)$ will be maximized over θ_1 by taking $\theta_1 = 0$. Therefore, the maximum of $\rho_X(2)$ is $\max\limits_{\theta_2>0} \dfrac{\theta_2}{1+\theta_2^2}$. Taking the derivative w.r.t. θ_2 and setting this to 0 gives

$$\frac{(1+\theta_2^2) - \theta_2 2\theta_2}{(1+\theta_2^2)^2} = 0 \iff 1+\theta_2^2 - 2\theta_2^2 = 0 \iff 1 = \theta_2^2$$

So $\theta_2 = +1$ (and $\theta_1 = 0$) maximizes $\rho_X(2)$. The maximum value is $\dfrac{1}{1+0^2+1^2} = \dfrac{1}{2}$. Therefore $|\rho_X(2)| \leq \frac{1}{2}$.

Now, consider $\rho_X(1)$. First, let us note that to maximize $\rho_X(1)$ over θ_1 and θ_2 we can assume that $\theta_2 \neq -1$ because when $\theta_2 = -1$, $\rho_X(1) = 0$ (which is clearly not the maximum).

For a fixed $\theta_2 \neq -1$, differentiating $\rho_X(1)$ w.r.t. θ_1 and setting to 0 gives

$$\frac{(1+\theta_1^2+\theta_2^2)(1+\theta_2) - \theta_1(1+\theta_2)2\theta_1}{(1+\theta_1^2+\theta_2^2)^2} = 0$$

$$\iff 1+\theta_1^2+\theta_2^2 = 2\theta_1^2 \quad \text{since } 1+\theta_2 \neq 0$$

$$\iff 1+\theta_2^2 = \theta_1^2$$

Thus, we get that $\theta_1 = \pm\sqrt{1+\theta_2^2}$ will potentially maximize $\rho_X(1)$ for a given value of θ_2. Let $\rho_X(1|\theta_2)$ be the maximum value of $\rho_X(1)$ for the given θ_2. Thus, we wish to maximize $\rho_X(1|\theta_2)$ over θ_2.

For a fixed θ_2, when we plug in the values $\theta_1 = \pm\sqrt{1+\theta_2^2}$ we get $\dfrac{\pm\sqrt{1+\theta_2^2}\,(1+\theta_2)}{1+(1+\theta_2^2)+\theta_2^2}$. We wish to maximize this over $\theta_2 \neq -1$. Rather than maximizing the above over θ_2, we will maximize the square of the above over θ_2, since the square is a monotone function so the value of θ_2 that maximizes the square also maximizes the above. Thus, we wish to maximize

$$(*)\quad \frac{(1+\theta_2^2)(1+\theta_2)^2}{(2(1+\theta_2^2))^2} = \frac{(1+\theta_2)^2}{4(1+\theta_2^2)} = \frac{(1+\theta_2^2)+2\theta_2}{4(1+\theta_2^2)} = \frac{1}{4} + \frac{\theta_2}{2(1+\theta_2^2)}$$

We have already computed that $\dfrac{\theta_2}{1+\theta_2^2}$ is maximized at $\theta_2 = 1$, and the maximum value of $\dfrac{\theta_2}{1+\theta_2^2}$ is $\frac{1}{2}$. Therefore, $(*)$ is maximized at $\theta_2 = 1$, and its maximum value is $\frac{1}{4} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$.
Since $(*)$ is the square of $\rho_X(1)$ at a given θ_2. Therefore, the maximum value of $\rho_X(1)^2$ is $\frac{1}{2}$, which gives that the maximum value of $|\rho_X(1)|$ is $\frac{1}{\sqrt{2}}$. Therefore, $|\rho_X(1)| \le \frac{1}{\sqrt{2}}$.

④ Let $\{X_t\}$ be an ARMA(1,1) process with parameters $\phi = \frac{1}{2}$ and $\theta = \frac{1}{2}$, and $\sigma^2 = 1$, i.e., $X_t - \frac{1}{2}X_{t-1} = Z_t + \frac{1}{2}Z_{t-1}$, where $\{Z_t\}$ is a zero-mean WN(1). The ACVF of the ARMA(1,1) process is

$$\gamma_X(0) = \sigma^2\left(1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right)$$

$$\gamma_X(1) = \sigma^2\left(\theta + \phi + \frac{(\theta+\phi)\phi}{1-\phi^2}\right)$$

$$\gamma_X(h) = \phi^{h-1}\gamma_X(1) \qquad h \geq 2$$

with $\phi = \frac{1}{2}$, $\theta = \frac{1}{2}$, $\sigma^2 = 1$

$$\gamma_X(0) = \frac{7}{3}$$

$$\gamma_X(1) = \frac{5}{3}$$

$$\gamma_X(h) = \left(\tfrac{1}{2}\right)^{h-1}\left(\tfrac{5}{3}\right)$$

(a) Give $P(X_2|X_1)$, the best linear predictor of X_2 in terms of X_1 (this is also denoted by P_1X_2). Compute the mean squared error of $P(X_2|X_1)$.

Sol. Recall, $P(X_2|X_1) = aX_1$, where a satisfies

$$\gamma_X(0)a = \gamma_X(1), \quad \text{or} \quad a = \frac{\gamma_X(1)}{\gamma_X(0)} = \frac{5/3}{7/3} = \frac{5}{7}.$$

So $P(X_2|X_1) = \frac{5}{7}X_1$. The minimum mean squared error is

$$E[(X_2 - P(X_2|X_1))^2] = \gamma_X(0) - \tfrac{5}{7}\gamma_X(1) = \tfrac{7}{3} - \left(\tfrac{5}{7}\right)\left(\tfrac{5}{3}\right) = \tfrac{24}{21} = \tfrac{8}{7}.$$

(b) Find $P(X_3|X_2,X_1)$. Compute the MSE of $P(X_3|X_2,X_1)$.

Sol. $P(X_3|X_2,X_1) = a_1X_2 + a_2X_1$, where $a = (a_1,a_2)^T$ satisfies

$$\begin{bmatrix}\gamma_X(0) & \gamma_X(1)\\ \gamma_X(1) & \gamma_X(0)\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}\gamma_X(1)\\ \gamma_X(2)\end{bmatrix} \iff \begin{bmatrix}7/3 & 5/3\\ 5/3 & 7/3\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}5/3\\5/6\end{bmatrix}$$

$$\iff \begin{bmatrix}14 & 10\\ 10 & 14\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$$

The solution is

$$\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}14 & 10\\ 10 & 14\end{bmatrix}^{-1}\begin{bmatrix}10\\5\end{bmatrix} = \frac{1}{96}\begin{bmatrix}14 & -10\\ -10 & 14\end{bmatrix}\begin{bmatrix}10\\5\end{bmatrix} = \frac{1}{96}\begin{bmatrix}90\\-30\end{bmatrix} = \begin{bmatrix}\frac{15}{16}\\ -\frac{5}{16}\end{bmatrix}.$$

Then $P(X_3|X_2,X_1) = \frac{15}{16}X_2 - \frac{5}{16}X_1$. The minimum MSE is

$$E[(X_3 - P(X_3|X_2,X_1))^2] = \frac{7}{3} - \left(\left(\tfrac{15}{16}\right)\left(\tfrac{5}{3}\right) - \left(\tfrac{5}{16}\right)\left(\tfrac{5}{6}\right)\right)$$

$$= \frac{224 - 150 + 25}{96} = \frac{99}{96} = \frac{33}{32} \approx 1.03125.$$

(c) Compute $P(X_n \mid X_2, X_1)$ and give the minimum MSE. Compute the limit of this MSE as $n \to \infty$.

Sol. $P(X_n \mid X_2, X_1) = a_1 X_2 + a_2 X_1$ where $a = (a_1, a_2)^T$ satisfies

$$\begin{bmatrix} 7/3 & 5/3 \\ 5/3 & 7/3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(n-2) \\ \gamma_X(n-1) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^{n-3}\left(\frac{5}{3}\right) \\ \left(\frac{1}{2}\right)^{n-2}\left(\frac{5}{3}\right) \end{bmatrix}$$

or
$$\begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Then
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \underbrace{\begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 5 \end{bmatrix}}_{\text{done in part (b)}} = \left(\frac{1}{2}\right)^{n-3} \begin{bmatrix} \frac{15}{16} \\ -\frac{5}{16} \end{bmatrix}$$

Then $P(X_n \mid X_2, X_1) = \left(\frac{1}{2}\right)^{n-3}\left(\frac{15}{16} X_2 - \frac{5}{16} X_1\right)$

The minimum MSE is

$$E\left[\left(X_n - P(X_n \mid X_2, X_1)\right)^2\right] = \gamma_X(0) - \left(\frac{1}{2}\right)^{2n-6}\left(\left(\frac{15}{16}\right)\left(\frac{5}{3}\right) - \left(\frac{5}{16}\right)\left(\frac{5}{6}\right)\right)$$
$$= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \frac{150-25}{96}$$
$$= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \frac{125}{96}.$$

As $n \to \infty$, $E\left[\left(X_n - P(X_n \mid X_2, X_1)\right)^2\right] \to \frac{7}{3}$.