Queen's University
Department of Mathematics and Statistics

STAT 464/864

Final Examination, April 19, 2020
Instructor: G. Takahara

INSTRUCTIONS:

- The exam has 5 questions, each worth 15 marks. STAT 464 students must do 4 of the 5 questions for a total of 60 marks and STAT 864 students must do all 5 questions for a total of 75 marks. Students in STAT 464 can choose any 4 of the 5 questions but must indicate which of the 4 solutions to mark if there is any ambiguity. The default will be to mark questions 1-4.
- The exam is a "take-home" 24 hour exam. The exam is open book. This means that you can use your notes, the textbook, and your computer.
- For each question, begin each solution at the start of a fresh page, and put your student number at the start of each solution.
- **The solutions are to be submitted through crowdmark.** You will receive an invitation from crowdmark shortly before the exam is posted on the course web page (so around 1:45pm on April 19).
- **The deadline for submission of your solutions is 3pm on April 20**. So you will have 24 hours from 2pm on April 19 to 2pm on April 20, and you will have one hour to prepare your solutions for upload to crowdmark.
- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED ON THE EXAM.** There is to be no collaboration in any form on any question on any part of the exam, either in person or remotely. All work on the exam must be completed *on your own.*

Instructions continued on page 2.

- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
- This material is copyrighted and is for the sole use of students registered in STAT 464/864 and writing this examination. This material shall not be distributed or disseminated. Failure to abide by these conditions is a breach of copyright and may also constitute a breach of academic integrity under the University Senates Academic Integrity Policy Statement.
- You may write your solutions in the space provided, continuing on your own paper if needed, or you may write your solutions using your own paper.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

Problem 1 [15] Problem 2 [15] Problem 3 [15]

Problem 4 [15] Problem 5 [15]

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Student Number

1(a) (i) Show that a causal filter $\{a_j\}_{j=0}^{\infty}$ will pass (unchanged) an arbitrary quadratic polynomial, say $m_t = c_0 + c_1 t + c_2 t^2$, if the following three conditions are satisfied: $\sum_{j=0}^{\infty} a_j = 1$, $\sum_{j=0}^{\infty} j a_j = 0$ and $\sum_{j=0}^{\infty} j^2 a_j = 0$. [4]

(ii) Find a causal filter $\{a_0, \ldots, a_6\}$, where $a_j \neq 0$ for $j = 0, \ldots, 6$, that passes arbitrary quadratic polynomials and eliminates seasonal components with period 5. *Hint:* Construct the filter in 2 steps: first eliminate the seasonal component then apply a causal filter of length 3, $\{b_0, b_1, b_2\}$, and use (i) to determine b_0, b_1, b_2. [6]

(b) Find a symmetric filter $\{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3\}$ where $a_j \neq 0$ for $j = -3, \ldots, 3$ and $a_{-j} = a_j$, that passes arbitrary cubic polynomials and eliminates seasonal components with period either 2 or 3. [5]

Student Number

2. Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$.

(a) Let $\{X_t\}$ be an MA(2) process with parameters θ_1, θ_2, and σ^2, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$. Find $\gamma(h)$ and $\rho(h)$, the autocovariance function and autocorrelation function, respectively, of $\{X_t\}$ in terms of θ_1, θ_2 and σ^2, for $h = 0, \pm 1, \pm 2, \ldots$. [5]

(b) Suppose $\sigma^2 = 1$ and let $\{Y_t\}$ be the MA(1) process $Y_t = Z_t + \frac{1}{2}Z_{t-1}$. Let $X_t = Y_t + aY_{t-1}$, where a is a constant. Show that $\{X_t\}$ is an MA(2) process and find the parameters θ_1 and θ_2 of the process in terms of a. Find a so that $\{X_t\}$ has autocorrelation function specified by $\rho(0) = 1$, $\rho(1) = -\frac{1}{6}$, and $\rho(2) = -\frac{1}{3}$. [10]

Student Number

3(a) For each of the following functions defined on the integers, explain why it cannot be the autocorrelation function of a stationary time series. [3]

(i) $f_1(h) = \begin{cases} 1 & \text{if } h = 0 \\ 1/h & \text{if } h \neq 0 \end{cases}$ (ii) $f_2(h) = 1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4}$.

(b) For each of the following functions defined on the integers, specify a stationary time series such that the given function is its autocorrelation function. [4]

(iii) $f_3(h) = (-1)^{|h|}$ (iv) $f_4(h) = \frac{1}{3}\left(1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4}\right)$.

(c) Let $\{X_t\}$ be an MA(2) process with parameters θ_1, θ_2, and σ^2, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\} \sim$WN$(0, \sigma^2)$. Let $\rho(h)$ denote the autocorrelation function of $\{X_t\}$. Give the largest possible values of $|\rho(1)|$ and $|\rho(2)|$ (and prove your answers). You may refer back to Problem 2(a). [8]

Student Number

4. Let $\{X_t\}$ be the ARMA(1,1) process with parameters $\phi = 1/2$, $\theta = 1/2$, and $\sigma^2 = 1$, i.e., $\{X_t\}$ satisfies $X_t - \frac{1}{2}X_{t-1} = Z_t + \frac{1}{2}Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0,1)$. The autocovariance function $\gamma(h)$ of $\{X_t\}$ is given in Example 3.2.1 on p.78 of the text (with ϕ, θ and σ^2 as specified in this problem). Use this to do the following parts (actual numbers should be given for the coefficients of the linear predictions and for the mean squared errors).

(a) Give $P(X_2 \mid X_1)$, the best linear predictor of X_2 in terms of X_1. Compute the mean squared error of $P(X_2 \mid X_1)$. [3]

(b) Give $P(X_3 \mid X_1, X_2)$, the best linear predictor of X_3 in terms of X_1 and X_2. Compute the mean squared error of $P(X_3 \mid X_1, X_2)$. [7]

(c) Give $P(X_n \mid X_1, X_2)$, the best linear predictor of X_n in terms of X_1 and X_2, for $n > 3$. Compute the mean squared error of $P(X_n \mid X_1, X_2)$. Compute the limit of this mean squared error as $n \to \infty$. [5]

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Student Number

5. Consider the ARMA(2,1) process $\{X_t\}$ satisfying

$$(1 - .5B + .04B^2)X_t = (1 + .25B)Z_t$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Find the coefficients ψ_j in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

Hint: Express $1 - .5z + .04z^2$ as $a(1 - bz)(1 - cz)$ by finding the roots of the quadratic and find a, b and c. Then go from there. [15]