Queen's University
Department of Mathematics and Statistics

# STAT 464/864

Final Examination Solutions, 2020
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**INSTRUCTIONS:**

- The exam has 5 questions, each worth 15 marks. STAT 464 students must do 4 of the 5 questions for a total of 60 marks and STAT 864 students must do all 5 questions for a total of 75 marks. Students in STAT 464 can choose any 4 of the 5 questions but must indicate which of the 4 solutions to mark if there is any ambiguity. The default will be to mark questions 1-4.
- The exam is a "take-home" 24 hour exam. The exam is open book. This means that you can use your notes, the textbook, and your computer.
- For each question, begin each solution at the start of a fresh page, and put your student number at the start of each solution.
- **The solutions are to be submitted through crowdmark.** You will receive an invitation from crowdmark shortly before the exam is posted on the course web page (so around 1:45pm on April 19).
- **The deadline for submission of your solutions is 3pm on April 20**. So you will have 24 hours from 2pm on April 19 to 2pm on April 20, and you will have one hour to prepare your solutions for upload to crowdmark.
- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED ON THE EXAM.** There is to be no collaboration in any form on any question on any part of the exam, either in person or remotely. All work on the exam must be completed *on your own*.

Instructions continued on page 2.

- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
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- You may write your solutions in the space provided, continuing on your own paper if needed, or you may write your solutions using your own paper.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

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|---|---|---|
| Problem 1 [15] | Problem 2 [15] | Problem 3 [15] |
| Problem 4 [15] | Problem 5 [15] | |

Student Number

**1(a)** (i) Show that a causal filter $\{a_j\}_{j=0}^{\infty}$ will pass (unchanged) an arbitrary quadratic polynomial, say $m_t = c_0 + c_1 t + c_2 t^2$, if the following three conditions are satisfied: $\sum_{j=0}^{\infty} a_j = 1$, $\sum_{j=0}^{\infty} j a_j = 0$ and $\sum_{j=0}^{\infty} j^2 a_j = 0$. [4]

*Solution:* Applying the filter to $\{m_t\}$ gives

$$\begin{aligned}\sum_{j=0}^{\infty} a_j m_{t-j} &= \sum_{j=0}^{\infty} a_j (c_0 + c_1(t-j) + c_2(t-j)^2) \\ &= (c_0 + c_1 t + c_2 t^2) \sum_{j=0}^{\infty} a_j - (c_1 + 2c_2 t) \sum_{j=0}^{\infty} j a_j + c_2 \sum_{j=0}^{\infty} j^2 a_j,\end{aligned}$$

from which we can easily see the sufficiency of the given conditions.

(ii) Find a causal filter $\{a_0, \ldots, a_6\}$, where $a_j \neq 0$ for $j = 0, \ldots, 6$, that passes arbitrary quadratic polynomials and eliminates seasonal components with period 5. *Hint:* Construct the filter in 2 steps: first eliminate the seasonal component then apply a causal filter of length 3, $\{b_0, b_1, b_2\}$, and use (i) to determine $b_0, b_1, b_2$. [6]

*Solution:* First apply the filter $1 + B + B^2 + B^3 + B^4$, which will eliminate the seasonal component of period 5. Then apply the length 3 filter $\{b_0, b_1, b_2\}$ to get the composite filter

$$(b_0 + b_1 B + b_2 B^2)(1 + B + B^2 + B^3 + B^4) = \sum_{j=0}^{6} a_j B^j,$$

where $a_0 = b_0$, $a_1 = b_0 + b_1$, $a_2 = a_3 = a_4 = b_0 + b_1 + b_2$, $a_5 = b_1 + b_2$, $a_6 = b_2$. Note that the seasonal component is still gone from the output of this filter. By part(i), this filter will pass a quadratic polynomial if the following conditions on $\{b_0, b_1, b_2\}$ are satisfied:

$$\begin{aligned}1 &= 5(b_0 + b_1 + b_2) \\ 0 &= 10b_0 + 15b_1 + 20b_2 \\ 0 &= 30b_0 + 55b_1 + 90b_2\end{aligned}$$

These equations have solution $(b_0, b_1, b_2)^T = (1, -1.2, 0.4)^T$. Therefore, the final filter is given by

$$\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\} = \{1, -.2, .2, .2, .2, -.8, .4\}.$$

**(b)** Find a symmetric filter $\{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3\}$ where $a_j \neq 0$ for $j = -3, \ldots, 3$ and $a_{-j} = a_j$, that passes arbitrary cubic polynomials and eliminates seasonal components with period either 2 or 3. [5]

*Solution:* The desired symmetric filter has 4 unknown coefficients: $a_0, a_1, a_2$ and $a_3$. To pass an arbitrary cubic polynomial these coefficients must satisfy

$$\begin{aligned} a_0 + 2a_1 + 2a_2 + 2a_3 &= 1 \quad (1) \\ a_1 + 4a_2 + 9a_3 &= 0 \quad (2) \end{aligned}$$

When applied to the seasonal component $\{s_t\}$ the output will be

$$\begin{aligned} & a_3 s_{t-3} + a_2 s_{t-2} + a_1 s_{t-1} + a_0 s_t + a_1 s_{t+1} + a_2 s_{t+2} + a_3 s_{t+3} \\ = & \begin{cases} (a_0 + 2a_3)s_t + (a_1 + a_2)s_{t-1} + (a_1 + a_2)s_{t-2} & \text{if period is 3} \\ (a_0 + 2a_2)s_t + 2(a_1 + a_3)s_{t-1} & \text{if period is 2} \end{cases} \end{aligned}$$

From the above, to eliminate a seasonal component with period 2 or 3, it is then sufficient to impose the further 2 constraints:

$$\begin{aligned} a_0 + 2a_3 &= a_1 + a_2 \quad (3) \\ a_0 + 2a_2 &= 2(a_1 + a_3) \quad (4) \end{aligned}$$

The solution to the equations (1)-(4) is given by

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 4 & 9 \\ 1 & -1 & -1 & 2 \\ 1 & -2 & 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.43055 \\ 0.29861 \\ 0.03472 \\ -0.04861 \end{bmatrix}$$

Thus, the desired filter is given by

$$\begin{aligned} & \{a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3\} \\ = & \{-0.04861, 0.03472, 0.29861, 0.43055, 0.29861, 0.03472, -0.04861\} \end{aligned}$$

Student Number

**2.** Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$.

**(a)** Let $\{X_t\}$ be an MA(2) process with parameters $\theta_1$, $\theta_2$, and $\sigma^2$, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$. Find $\gamma(h)$ and $\rho(h)$, the autocovariance function and autocorrelation function, respectively, of $\{X_t\}$ in terms of $\theta_1$, $\theta_2$ and $\sigma^2$, for $h = 0, \pm 1, \pm 2, \ldots$. [5]

*Solution:* We have

$$\begin{aligned}
\text{Var}(X_t) &= \sigma^2(1 + \theta_1^2 + \theta_2^2) \\
\text{Cov}(X_t, X_{t-1}) &= \text{Cov}(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-1} + \theta_1 Z_{t-2} + \theta_2 Z_{t-3}) = \sigma^2\theta_1(1 + \theta_2) \\
\text{Cov}(X_t, X_{t-2}) &= \text{Cov}(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t-2} + \theta_1 Z_{t-3} + \theta_2 Z_{t-4}) = \sigma^2\theta_2.
\end{aligned}$$

Then we have that the autocovariance function is given by

$$\gamma(h) = \begin{cases} \sigma^2(1 + \theta_1^2 + \theta_2^2) & \text{if } h = 0 \\ \sigma^2\theta_1(1 + \theta_2) & \text{if } h = \pm 1 \\ \sigma^2\theta_2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

and the autocorrelation function is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & \text{if } h = 0 \\ \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} & \text{if } h = \pm 1 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } h = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

**(b)** Suppose $\sigma^2 = 1$ and let $\{Y_t\}$ be the MA(1) process $Y_t = Z_t + \frac{1}{2}Z_{t-1}$. Let $X_t = Y_t + aY_{t-1}$, where $a$ is a constant. Show that $\{X_t\}$ is an MA(2) process and find the parameters $\theta_1$ and $\theta_2$ of the process in terms of $a$. Find $a$ so that $\{X_t\}$ has autocorrelation function specified by $\rho(0) = 1$, $\rho(1) = -\frac{1}{6}$, and $\rho(2) = -\frac{1}{3}$. [10]

*Solution:* We have

$$\begin{aligned} X_t = Y_t + aY_{t-1} &= Z_t + \frac{1}{2}Z_{t-1} + a\left(Z_{t-1} + \frac{1}{2}Z_{t-2}\right) \\ &= Z_t + \left(\frac{1}{2} + a\right)Z_{t-1} + \frac{a}{2}Z_{t-2}. \end{aligned}$$

Thus, $\{X_t\}$ is an MA(2) process with parameters $\theta_1 = \frac{1}{2} + a$ and $\theta_2 = \frac{a}{2}$. To find $a$ so that $\rho(1) = -\frac{1}{6}$ and $\rho(2) = -\frac{1}{3}$, from part(a) the coefficient $a$ should satisfy

$$\frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} = -\frac{1}{6} \quad \text{and} \quad \frac{\theta_2}{1+\theta_1^2+\theta_2^2} = -\frac{1}{3}$$

In terms of $a$, these equations can be written as

$$\frac{2+5a+2a^2}{5+4a+5a^2} = -\frac{1}{6} \quad \text{and} \quad \frac{2a}{5+4a+5a^2} = -\frac{1}{3}. \tag{5}$$

The second equation above can be written as $\frac{5a}{2} + 2 + \frac{5}{2a} = -3$, or $a^2 + 2a + 1 = 0$. This quadratic equation has only a single root given by $a = -1$, and one can check that this solves both of the equations in (5).

Student Number

**3(a)** For each of the following functions defined on the integers, explain why it cannot be the autocorrelation function of a stationary time series. [3]

(i) $f_1(h) = \begin{cases} 1 & \text{if } h = 0 \\ 1/h & \text{if } h \neq 0 \end{cases}$ (ii) $f_2(h) = 1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4}$.

*Solution:*

(i) $f_1(h)$ is not symmetric about 0 and so cannot be an autocorrelation function.

(ii) $f_2(0) = 3$ and so cannot be an autocorrelation function.

**(b)** For each of the following functions defined on the integers, specify a stationary time series such that the given function is its autocorrelation function. [4]

(iii) $f_3(h) = (-1)^{|h|}$ (iv) $f_4(h) = \frac{1}{3}\left(1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4}\right)$.

*Solution:*

(iii) Take $X_t = (-1)^t X$, where $X$ is any random variable with unit variance.

(iv) Take $X_t = A_0 + A_1 \cos\frac{\pi t}{2} + A_2 \sin\frac{\pi t}{2} + A_3 \cos\frac{\pi t}{4} + A_4 \sin\frac{\pi t}{4}$, where $A_0, \ldots, A_4$ are independent, zero-mean, unit variance random variables. Then $\mathrm{Var}(X_t) = 3$ for all $t$ and

$$\begin{aligned}
& \mathrm{Cov}(X_t, X_{t-h}) \\
= \;& E\left[\left(A_0 + A_1 \cos\frac{\pi t}{2} + A_2 \sin\frac{\pi t}{2} + A_3 \cos\frac{\pi t}{4} + A_4 \sin\frac{\pi t}{4}\right)\right. \\
& \left.\times \left(A_0 + A_1 \cos\frac{\pi(t-h)}{2} + A_2 \sin\frac{\pi(t-h)}{2} + A_3 \cos\frac{\pi(t-h)}{4} + A_4 \sin\frac{\pi(t-h)}{4}\right)\right] \\
= \;& 1 + \cos\frac{\pi t}{2}\cos\frac{\pi(t-h)}{2} + \sin\frac{\pi t}{2}\sin\frac{\pi(t-h)}{2} + \cos\frac{\pi t}{4}\cos\frac{\pi(t-h)}{4} + \sin\frac{\pi t}{4}\sin\frac{\pi(t-h)}{4} \\
= \;& 1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4},
\end{aligned}$$

using the identity $\cos(a-b) = \cos a \cos b + \sin a \sin b$. Dividing this by $\mathrm{Var}(X_t) = 3$ gives $f_4(h)$.

**(c)** Let $\{X_t\}$ be an MA(2) process with parameters $\theta_1$, $\theta_2$, and $\sigma^2$, i.e., $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, where $\{Z_t\} \sim$WN$(0, \sigma^2)$. Let $\rho(h)$ denote the autocorrelation function of $\{X_t\}$. Give the largest possible values of $|\rho(1)|$ and $|\rho(2)|$ (and prove your answers). You may refer back to Problem 2(a). [8]

*Solution:* From the solution to Problem 2(a), we have

$$\rho(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \quad \text{and} \quad \rho(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}.$$

Considering $\rho(2)$ first, it is clear that $|\rho(2)|$ is maximized by setting $\theta_1 = 0$ and maximizing $\frac{\theta_2}{1+\theta_2^2}$ over $\theta_2 > 0$. This is maximized at $\theta_2 = 1$ and so the maximum value of $|\rho(2)|$ is $1/2$, and this is achieved at $\theta_1 = 0$ and $\theta_2 = 1$. For $\rho(1)$, note that when $\theta_2 = -1$ then $\rho(1) = 0$ for all $\theta_1$, so $\rho(1)$ is not maximized at $(\theta_1, 0)$ for any $\theta_1$. For any fixed value of $\theta_2 \neq -1$ we can find the value of $\theta_1$ (in terms of $\theta_2$) that maximizes $\rho(1)$ by differentiating with respect to $\theta_1$ and setting the derivative to 0. This gives

$$(1+\theta_1^2+\theta_2^2)(1+\theta_2) - 2\theta_1^2(1+\theta_2) = 0 \Leftrightarrow 1+\theta_2^2 = \theta_1^2.$$

So we get $\theta_1 = \pm\sqrt{1+\theta_2^2}$, and plugging this into $\rho(1)$, we can write

$$|\rho(1 \mid \theta_2)| = \frac{\sqrt{1+\theta_2^2}|1+\theta_2|}{2(1+\theta_2^2)} = \frac{|1+\theta_2|}{2\sqrt{1+\theta_2^2}},$$

for $\theta_2 \neq -1$, where $|\rho(1 \mid \theta_2)|$ is the maximum value $|\rho(1)|$ for fixed $\theta_2$. The maximum value of $|\rho(1)|$ is then obtained by maximizing $|\rho(1 \mid \theta_2)|$ over $\theta_2$. Rather than maximizing $|\rho(1 \mid \theta_2)|$ we maximize $\rho(1 \mid \theta_2)^2$ since both are maximized at the same value of $\theta_2$ since $f(x) = x^2$ is a monotone increasing function. We have

$$\rho(1 \mid \theta_2)^2 = \frac{(1+\theta_2)^2}{4(1+\theta_2^2)} = \frac{1+\theta_2^2+2\theta_2}{4(1+\theta_2^2)} = \frac{1}{4} + \frac{1}{2}\frac{\theta_2}{1+\theta_2^2}.$$

This is maximized at $\theta_2 = 1$, and so plugging this into $|\rho(1 \mid \theta_2)|$ we get

$$\max_{\theta_1,\theta_2\in\mathbb{R}} |\rho(1)| = \max_{\theta_2\in\mathbb{R}} |\rho(1 \mid \theta_2)| = |\rho(1 \mid 1)| = \frac{1}{\sqrt{2}}.$$

To summarize,

$$\max_{\theta_1,\theta_2\in\mathbb{R}} |\rho(1)| = \frac{1}{\sqrt{2}} \quad \text{and} \quad \max_{\theta_1,\theta_2\in\mathbb{R}} |\rho(2)| = \frac{1}{2}.$$

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Student Number

**4.** Let $\{X_t\}$ be the ARMA(1,1) process with parameters $\phi = 1/2$, $\theta = 1/2$, and $\sigma^2 = 1$, i.e., $\{X_t\}$ satisfies $X_t - \frac{1}{2}X_{t-1} = Z_t + \frac{1}{2}Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0,1)$. The autocovariance function $\gamma(h)$ of $\{X_t\}$ is given in Example 3.2.1 on p.78 of the text (with $\phi$, $\theta$ and $\sigma^2$ as specified in this problem). Use this to do the following parts (actual numbers should be given for the coefficients of the linear predictions and for the mean squared errors).

**(a)** Give $P(X_2 \mid X_1)$, the best linear predictor of $X_2$ in terms of $X_1$. Compute the mean squared error of $P(X_2 \mid X_1)$. [3]

*Solution:* We first note that with $\phi = 1/2$, $\theta = 1/2$, and $\sigma^2 = 1$, the autocovariance function of $\{X_t\}$ is given by $\gamma(0) = \frac{7}{3}$, $\gamma(1) = \frac{5}{3}$, and $\gamma(h) = \frac{5}{3}\left(\frac{1}{2}\right)^{h-1}$ for $h \geq 2$. Then

$$P(X_2 \mid X_1) = \frac{\gamma(1)}{\gamma(0)}X_1 = \frac{5}{7}X_1.$$

The minimum mean squared error is given by

$$E[(X_2 - P(X_2 \mid X_1))^2] = \gamma(0) - \frac{5}{7}\gamma(1) = \frac{7}{3} - \left(\frac{5}{7}\right)\left(\frac{5}{3}\right) = \frac{24}{21} = \frac{8}{7}.$$

**(b)** Give $P(X_3 \mid X_1, X_2)$, the best linear predictor of $X_3$ in terms of $X_1$ and $X_2$. Compute the mean squared error of $P(X_3 \mid X_1, X_2)$. [7]

*Solution:* $P(X_3 \mid X_1, X_2)$ is given by $a_1 X_2 + a_2 X_1$, where $a = (a_1, a_2)^T$ satisfies

$$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix} \Leftrightarrow \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The solution is given by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 14 & -10 \\ -10 & 14 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 90 \\ -30 \end{bmatrix} = \left( \frac{15}{16}, -\frac{5}{16} \right)^T .$$

Thus,

$$P(X_3 \mid X_1, X_2) = \frac{15}{16} X_2 - \frac{5}{16} X_1 = 0.9375 X_2 - 0.3125 X_1.$$

The minimum mean squared error is given by

$$\begin{aligned} E[(X_3 - P(X_3 \mid X_1, X_2))^2] &= \gamma(0) - a_1 \gamma(1) - a_2 \gamma(2) \\ &= \frac{7}{3} - \left(\frac{15}{16}\right)\left(\frac{5}{3}\right) + \left(\frac{5}{16}\right)\left(\frac{5}{6}\right) \\ &= \frac{224 - 150 + 25}{96} = \frac{99}{96} = \frac{33}{32} \approx 1.03125. \end{aligned}$$

**(c)** Give $P(X_n \mid X_1, X_2)$, the best linear predictor of $X_n$ in terms of $X_1$ and $X_2$, for $n > 3$. Compute the mean squared error of $P(X_n \mid X_1, X_2)$. Compute the limit of this mean squared error as $n \to \infty$. [5]

*Solution:* $P(X_n \mid X_1, X_2)$ is given by $a_1 X_2 + a_2 X_1$, where $a = (a_1, a_2)^T$ satisfies

$$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma(n-2) \\ \gamma(n-1) \end{bmatrix} \Leftrightarrow \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

From part(b), the solution is given by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{2}\right)^{n-3} \left(\frac{15}{16}, -\frac{5}{16}\right)^T,$$

and so

$$P(X_n \mid X_1, X_2) = \left(\frac{1}{2}\right)^{n-3} \left[\frac{15}{16} X_2 - \frac{5}{16} X_1\right] = \left(\frac{1}{2}\right)^{n-3} (0.9375 X_2 - 0.3125 X_1).$$

The minimum mean squared error is given by

$$\begin{aligned} E[(X_n - P(X_3 \mid X_1, X_2))^2] &= \gamma(0) - a_1 \gamma(n-2) - a_2 \gamma(n-1) \\ &= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \left[\left(\frac{15}{16}\right)\left(\frac{5}{3}\right) - \left(\frac{5}{16}\right)\left(\frac{5}{6}\right)\right] \\ &= \frac{7}{3} - \left(\frac{1}{2}\right)^{2n-6} \frac{125}{96}. \end{aligned}$$

This converges to $\gamma(0) = \frac{7}{3}$ as $n \to \infty$.

Student Number

**5.** Consider the ARMA(2,1) process $\{X_t\}$ satisfying

$$(1 - .5B + .04B^2)X_t = (1 + .25B)Z_t$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Find the coefficients $\psi_j$ in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

*Hint:* Express $1 - .5z + .04z^2$ as $a(1 - bz)(1 - cz)$ by finding the roots of the quadratic and find $a$, $b$ and $c$. Then go from there. [15]

*Solution:* Following the hint we find the roots of the quadratic polynomial $1 - .5z + .04z^2$, which are given by $\frac{.5 \pm \sqrt{.5^2 - 4(.04)}}{2(.04)} = \frac{.5 \pm .3}{.08} = 10$ or $2.5$. Thus, $1 - .5z + .04z^2 = \frac{1}{25}(10 - z)(2.5 - z) = (1 - .1z)(1 - .4z)$. Then we can write

$$X_t = \frac{1 + .25B}{(1 - .1B)(1 - .4B)} Z_t$$

and it remains to expand the operator $\frac{1+.25B}{(1-.1B)(1-.4B)}$ in a Taylor series expansion. Writing

$$\frac{1}{1 - .1B} \times \frac{1}{1 - .4B} = \sum_{j=0}^{\infty} (.1B)^j \sum_{k=0}^{\infty} (.4B)^k,$$

we have

$$\begin{aligned}
\frac{1 + .25B}{(1 - .1B)(1 - .4B)} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (.1B)^j (.4B)^k (1 + .25B) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (.1)^j (.4)^k B^{j+k} + .25 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (.1)^j (.4)^k B^{j+k+1} \\
&= 1 + \sum_{i=1}^{\infty} \left( \sum_{j=0}^{i} (.1)^j (.4)^{i-j} + .25 \sum_{j=0}^{i-1} (.1)^j (.4)^{i-j-1} \right) B^i \\
&= 1 + \sum_{i=1}^{\infty} \left( .4^i \frac{1 - .25^{i+1}}{1 - .25} + .25(.4)^{i-1} \frac{1 - .25^i}{1 - .25} \right) B^i \\
&= 1 + \sum_{i=1}^{\infty} \left( \frac{4}{3}(.4)^i - \frac{1}{3}(.1)^i + \frac{5}{6}(.4)^i - \frac{5}{6}(.1)^i \right) B^i \\
&= 1 + \sum_{i=1}^{\infty} \left( \frac{13}{6}(.4)^i - \frac{7}{6}(.1)^i \right) B^i.
\end{aligned}$$

So $\psi_0 = 1$ and $\psi_j = \frac{1}{6}(13(.4)^j - 7(.1)^j)$ for $j \geq 1$.