

Queen's University
Department of Mathematics and Statistics

STAT 464/864

Final Examination, April 26, 2021

Instructor: G. Takahara

INSTRUCTIONS:

- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED ON THE EXAM.**

There is to be no collaboration in any form on any question on any part of the exam, either in person or remotely. All work on the exam must be completed *on your own*. Any suspicion of collaboration will be flagged by me and in this case the default will be that you will receive 0 on the question. You can appeal later if you think I was wrong. If, upon further investigation, I still conclude that there was collaboration then I will pursue the consequences of academic dishonesty, which can ultimately mean withdrawal from the university, quite a bit more vigorously.

- The exam has 5 questions, each worth 15 marks. STAT 464 students must do question 1-4 for a total of 60 marks and STAT 864 students must do all 5 questions for a total of 75 marks. Part marks are shown to the right.
- The exam is a “take-home” 24 hour exam. The exam is open book. This means that you can use your notes, the textbook, and your computer.
- For each question, write your solution using your own paper. You may use a tablet if that is more convenient for you. Begin each solution at the start of a fresh page, and put your student number at the start of each solution.
- **The solutions are to be submitted through crowdmark.** You will receive an invitation from crowdmark shortly before the exam is emailed to you.
- **The deadline for submission of your solutions is 9am on April 27, Kingston time.**

Instructions continued on page 2.

- “The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer.”
- This material is copyrighted and is for the sole use of students registered in STAT 464 and writing this examination. This material shall not be distributed or disseminated. Failure to abide by these conditions is a breach of copyright and may also constitute a breach of academic integrity under the University Senates Academic Integrity Policy Statement.
- **SHOW YOUR WORK CLEARLY.** Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

 Student Number

1 (a) Let $X_t = s_t^{(2)} + s_t^{(3)} + s_t^{(4)}$, where $\{s_t^{(2)}\}$, $\{s_t^{(3)}\}$, and $\{s_t^{(4)}\}$, are seasonal components with periods 2, 3, and 4, respectively. Find a linear filter $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ (i.e., a_0, \dots, a_5 are all nonzero and are the only nonzero coefficients of the filter) such that the output of the filter when applied to $\{X_t\}$ is a constant for all t . [8]

(b) Let d_1 and d_2 be positive integers, with $d_1 \neq d_2$. Let $\{s_t^{(d_1)}\}$ and $\{s_t^{(d_2)}\}$ be seasonal components with periods d_1 and d_2 , respectively. Let $X_t = s_t^{(d_1)} + s_t^{(d_2)}$ and let $Y_t = \nabla_{d_1} X_t$. For each of the following statements, say whether it is **TRUE** or **FALSE**. If **TRUE** prove the statement, and if **FALSE** give a counterexample.

(i) $\nabla_{d_1 d_2} X_t$ is a constant. [2]

(ii) $\nabla_{d_1 + d_2} X_t$ is a constant. [2]

(iii) $\nabla_{d_2} Y_t$ is a constant. [3]

2 (a) Let $\{Y_t\}$ be a weakly stationary process and let $X_t = B^r Y_t$, where B is the backshift operator and r is a positive integer. Do $\{Y_t\}$ and $\{X_t\}$ have the same ACVF? If true, prove your answer. If false, give a counterexample. [2]

(b) Let $\{Y_t\}$ be a weakly stationary process and let $X_t = \nabla_r Y_t$, where ∇_r is the lag r difference operator and r is a positive integer. Do $\{Y_t\}$ and $\{X_t\}$ have the same ACVF? If true, prove your answer. If false, give a counterexample. [3]

For parts (c) and (d) below, use the following notation and definitions. Let $\{Z_t\}$ be a zero-mean $WN(\sigma^2)$ process and let $\{s_t\}$ be a seasonal component with period $d \geq 2$ (note that this implies that $s_t = c$ for all t , where c is a constant, is not allowed). Let $W_t = s_t Z_t$.

(c) Show that $\{W_t\}$ is not stationary. [2]

(d) Suppose d is odd, say $d = 2q + 1$, where q is a positive integer. Also assume that $s_{d-1} \neq 0$. Let $X_t = \sum_{j=0}^{2q} s_j Z_{t-q+j}$. Show that $\{X_t\}$ is stationary and compute its ACVF and ACF. [8]

3. Let $\{X_t\}$ be an MA(2) process with parameters θ_1 , -1 , and σ^2 (i.e., $X_t = Z_t + \theta_1 Z_{t-1} - Z_{t-2}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process).

(a) What are the approximate variances of the sample autocorrelations $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ based on X_1, \dots, X_n for n large? Express your answers in terms of θ_1 and simplify as much as possible. What are these approximate variances when $\theta_1 = 0$? [10]

(b) What is the approximate covariance and correlation between the sample autocorrelations $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ based on X_1, \dots, X_n for n large? [3]

(c) What is the approximate variance of $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ for large n ? [2]

4.(a) Let $\{X_t\}$ be a zero-mean stationary process. Let ρ_1 , ρ_2 , and ρ_3 denote the ACF of $\{X_t\}$ at lags 1, 2, and 3, respectively. Find $P(X_2 \mid X_1)$, $P(X_3 \mid X_2, X_1)$, and $P(X_2 \mid X_3, X_1)$, expressing the coefficients in these predictions in terms of ρ_1 , ρ_2 , and ρ_3 . [7]

(b) Let $\{X_t\}$ be an MA(2) process with parameters θ_1 , -1 , and σ^2 (i.e., $X_t = Z_t + \theta_1 Z_{t-1} - Z_{t-2}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process). Find $P(X_4 \mid X_3, X_2, X_1)$, $P(X_4 \mid X_2, X_1)$, and $P(X_4 \mid X_1)$ and the mean squared errors of each of these predictions in terms of the parameters θ_1 and σ^2 . [8]

*5. Let $X_t = \phi X_{t-1} + \sum_{j=0}^q \theta^j Z_{t-j}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, $|\phi| < 1$, $|\theta| < 1$, $\theta \neq \phi$, $\phi \neq 0$, $\theta \neq 0$, and $q \geq 1$.

(a) Find the coefficients ψ_k in the representation $X_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}$. You must show your work in deriving the ψ_k . The ψ_k must be expressed in terms of ϕ and θ only and be as simplified as possible (there should not be any sums in your expression for ψ_k). [8]

(b) For $q = 2$ find the ACVF, $\gamma_X(h)$, of $\{X_t\}$. [7]