Queen's University
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Final Examination Solutions, Fall 2021

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1 In the following parts, ∇_d denotes the lag-d difference operator.

(a) Suppose $\{X_t\}$ has 2 seasonal components, one of period 12 and one of period 28. What is the smallest d such that $\{\nabla_d X_t\}$ will have no seasonal components? More generally, suppose $\{X_t\}$ has k seasonal components of periods $d_1, \ldots, d_k$. What is the smallest d such that $\{\nabla_d X_t\}$ will have no seasonal components? [5]

Solution: If $\{X_t\}$ has 2 seasonal components of periods 12 and 28, the smallest d such that $\{\nabla_d X_t\}$ will have no seasonal components is $d = 84$. More generally, if $\{X_t\}$ has k seasonal components of periods $d_1, \ldots, d_k$, the smallest d such that $\{\nabla_d X_t\}$ will have no seasonal components is $\operatorname{lcm}(d_1, \ldots, d_k)$, where lcm is the least common multiple.

(b) Suppose $\{X_t\}$ has 2 seasonal components, one of period 5 and one of period 7, and also a quadratic polynomial trend. Give a causal filter with no more than three nonzero coefficients such that the output of the filter applied to $\{X_t\}$ will have no trend and no seasonal components. [5]

Solution: From part(a), if we apply the filter ∇_{35} then $\{X_t\}$ will have no seasonal components. If we apply it again then the quadratic trend will be eliminated, and the resulting filter will have 3 nonzero coefficients:

$$\begin{aligned}
\nabla_{35}^2 X_t &= \nabla_{35}(X_t - X_{t-35}) \\
&= X_t - X_{t-35} - (X_{t-35} - X_{t-70}) \\
&= X_t - 2X_{t-35} + X_{t-70}.
\end{aligned}$$

So, the filter is given by $\{a_0, a_{35}, a_{70}\} = \{1, -2, 1\}$ and $a_j = 0$ for $j \neq 1, 35, 70$.

2 Let $\{X_t\}$ be a zero-mean stationary process, for $t \in \mathbb{Z}$, with ACF $\gamma_X(h)$. In each of the following parts, state whether the process $\{Y_t\}$ is necessarily stationary or not. If it is not necessarily stationary prove it. If it is necessarily stationary give the ACF of $\{Y_t\}$ in terms of the ACF of $\{X_t\}$: [10]

(a) $Y_t = (-1)^t X_t$; (b) $Y_t = X_{|t|}$; (c) $Y_t = X_{kt}$, where $k > 1$ is an integer; (d) $Y_t = X_{t^3}$.

Solution:

(a) For $Y_t = (-1)^t X_t$ we have $\operatorname{Cov}(Y_t, Y_{t+h}) = \operatorname{Cov}((-1)^t X_t, (-1)^{t+h} X_{t+h}) = (-1)^h \gamma_X(h)$. This does not depend on t so $\{Y_t\}$ is stationary with ACVF $\gamma_Y(h) = (-1)^h \gamma_X(h)$.

(b) The process $Y_t = X_{|t|}$ is not stationary. For example, let $\{X_t\}$ be zero mean white noise with variance 1. For $h = 2$ we have $\text{Cov}(Y_0, Y_2) = \text{Cov}(X_0, X_2) =$. However, $\text{Cov}(Y_{-1}, Y_1) = \text{Cov}(X_1, X_1) = 1$. Therefore, the covariance function of $\{Y_t\}$ at lag 2 changes depending on t.

(c) For $Y_t = X_{kt}$ we have $\text{Cov}(Y_t, Y_{t+h}) = \text{Cov}(X_{tk}, X_{(t+h)k}) = \gamma_X(hk)$, which does not depend on t. So $\{Y_t\}$ is stationary with ACVF $\gamma_Y(h) = \gamma_X(hk)$.

(d) The process $Y_t = X_{t^3}$ is not stationary. For example, at lag $h = 1$, we have that $\text{Cov}(Y_t, Y_{t+1}) = \text{Cov}(X_{t^3}, X_{(t+1)^3})$. At $t = 1$ this is $\text{Cov}(X_1, X_8) = \gamma_X(7)$ while at $t = 2$ this is $\text{Cov}(X_8, X_{27}) = \gamma_X(19)$, which may be different from $\gamma_X(7)$. Therefore, the covariance function of $\{Y_t\}$ at lag 1 changes depending on t.

3. Let $\{X_t\}$ be a stationary process. Compute $P(X_3 \mid X_2)$ and $P(X_1 \mid X_2)$ and show that the correlation between $X_3 - P(X_3 \mid X_2)$ and $X_1 - P(X_1 \mid X_2))$ is equal to the coefficient of X_1 in $P(X_3 \mid X_2, X_1)$. [10]

Solution: Let ρ denote the lag 1 correlation of $\{X_t\}$, i.e., $\rho = \rho_X(1)$, where $\rho_X(\cdot)$ is the ACF of $\{X_t\}$. Then $P(X_3 \mid X_2) = P(X_1 \mid X_2) = \rho X_2$ (easy to check). So,

$$\begin{aligned}
\rho(X_3 - P(X_3 \mid X_2), X_1 - P(X_1 \mid X_2)) &= \frac{\text{Cov}(X_3 - \rho X_2, X_1 - \rho X_2)}{\sqrt{\text{Var}(X_3 - \rho X_2)\text{Var}(X_1 - \rho X_2)}} \\
&= \frac{\gamma_X(2) - 2\rho\gamma_X(1) + \rho^2\gamma_X(0)}{(1+\rho^2)\gamma_X(0) - 2\rho\gamma_X(1)} \\
&= \frac{\rho_X(2) - 2\rho^2 + \rho^2}{1 + \rho^2 - 2\rho^2} = \frac{\rho_X(2) - \rho^2}{1 - \rho^2},
\end{aligned}$$

where $\gamma_X(\cdot)$ is the ACVF of $\{X_t\}$. On the other hand, $P(X_3 \mid X_2, X_1) = a_1 X_2 + a_2 X_1$, where $(a_1, a_2)^T$ satisfies

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \end{bmatrix}.$$

The fist equation gives $a_1 = \frac{\gamma_X(1) - a_2\gamma_X(1)}{\gamma_X(0)} = \rho(1 - a_2)$. Plugging this into the second equation gives

$$\gamma_X(1)\rho(1 - a_2) + \gamma_X(0)a_2 = \gamma_X(2) \Rightarrow \rho^2(1 - a_2) + a_2 = \rho_X(2) \Rightarrow a_2 = \frac{\rho_X(2) - \rho^2}{1 - \rho^2},$$

which is the same as the correlation between $X_3 - P(X_3 \mid X_2)$ and $X_1 - P(X_1 \mid X_2)$, as desired.

4. Let $\{X_t\}$ be the AR(2) process satisfying

$$X_t - \phi X_{t-1} - \phi X_{t-2} = Z_t, \tag{1}$$

where $\{Z_t\}$ is a zero-mean WN(σ^2) process. You may assume that ϕ is such that this AR(2) process is causal. By multiplying Eq.(1) by X_{t-k} and taking expectations, for $k = 1, 2, 3, 4, 5$, compute the ACF of $\{X_t\}$ at lags 1,2,3,4,5 in terms of ϕ. [10]

Solution: Since $\{X_t\}$ is causal, Z_t is uncorrelated with X_{t-k} for $k \geq 1$, multiplying the RHS of (1) by X_{t-k} and taking expectation gives 0 if $k \geq 1$. Multiplying both sides of (1) by X_{t-k} and taking expectation gives the equations

$$\begin{aligned}
\gamma_X(1) - \phi\gamma_X(0) - \phi\gamma_X(1) &= 0 \quad (k = 1) \\
\gamma_X(k) - \phi\gamma_X(k-1) - \phi\gamma_X(k-2) &= 0 \quad (k = 2, 3, 4, 5),
\end{aligned}$$

where $\gamma_X(\cdot)$ is the ACVF of $\{X_t\}$. Dividing through by $\gamma_X(0)$ gives

$$\begin{aligned}
\rho_X(1) - \phi - \phi\rho_X(1) &= 0 \\
\rho_X(k) - \phi\rho_X(k-1) - \phi\rho_X(k-2) &= 0 \quad (k = 2, 3, 4, 5),
\end{aligned}$$

where $\rho_X(\cdot)$ is the ACF of $\{X_t\}$. Solving, we get

$$\begin{aligned}
\rho_X(1) &= \frac{\phi}{1-\phi} \\
\rho_X(2) &= \phi\left(1 + \frac{\phi}{1-\phi}\right) = \frac{\phi}{1-\phi} \\
\rho_X(3) &= \phi\left(\frac{\phi}{1-\phi} + \frac{\phi}{1-\phi}\right) = \frac{2\phi^2}{1-\phi} \\
\rho_X(4) &= \phi\left(\frac{2\phi^2}{1-\phi} + \frac{\phi}{1-\phi}\right) = \frac{2\phi^3 + \phi^2}{1-\phi} \\
\rho_X(5) &= \phi\left(\frac{2\phi^3 + \phi^2}{1-\phi} + \frac{2\phi^2}{1-\phi}\right) = \frac{2\phi^4 + 3\phi^3}{1-\phi}.
\end{aligned}$$

*5. Let $\{X_t\}$ be the AR(2) process satisfying $X_t - \phi X_{t-1} - \phi X_{t-2} = Z_t$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process. For what values of $\phi \in \mathbb{R}$ is the process $\{X_t\}$ causal? [10]

Solution: First, if $\phi = 0$ then $X_t = Z_t$ and the process is trivially causal. For $\phi \neq 0$, the AR polynomial is $\phi(z) = 1 - \phi z - \phi z^2$. We wish to find all values of $\phi \neq 0$ such that both roots of this polynomial are outside the unit circle. From the quadratic formula, the two roots are given by

$$\frac{\phi \pm \sqrt{\phi^2 + 4\phi}}{-2\phi} = -\frac{1}{2} \pm \frac{\sqrt{\phi^2 + 4\phi}}{\phi}.$$

For $\phi > 0$, both roots are real and $\frac{\sqrt{\phi^2+4\phi}}{\phi} = \sqrt{1 + \frac{4}{\phi}}$. This is a strictly decreasing function of ϕ and for both roots to be outside the unit circle we require $\sqrt{1 + \frac{4}{\phi}} > \frac{3}{2}$, or $1 + \frac{4}{\phi} > \frac{9}{4}$, or $\phi < \frac{16}{5}$. If $\phi \in (-4, 0)$ then the two roots are complex conjugates and the squared magnitude of each root is $\frac{1}{4} + \frac{|\phi^2+4\phi|}{\phi^2} = \frac{1}{4} + \left|1 + \frac{4}{\phi}\right|$. So we need $\left|1 + \frac{4}{\phi}\right| > \frac{3}{4}$, or $\frac{4}{\phi} < -\frac{7}{4}$, or $\phi \in \left(-\frac{16}{7}, 0\right)$. If $\phi \leq -4$ then both roots are real again and $\frac{\sqrt{\phi^2+4\phi}}{\phi} = -\sqrt{1 - \frac{4}{|\phi|}}$. The two roots in this case will be $-\frac{1}{2} \pm \sqrt{1 - \frac{4}{|\phi|}}$. The root $-\frac{1}{2} + \sqrt{1 - \frac{4}{|\phi|}}$ will always be in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right)$ for $\phi \leq -4$, and so $\{X_t\}$ will not be causal in this case. To summarize, the process $\{X_t\}$ will be causal for $\phi \in \left(-\frac{16}{7}, \frac{16}{5}\right)$.