Queen's University
Department of Mathematics and Statistics

STAT 464/864

Final Examination, Solutions, 2021
Instructor: G. Takahara

INSTRUCTIONS:

- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED ON THE EXAM.** There is to be no collaboration in any form on any question on any part of the exam, either in person or remotely. All work on the exam must be completed *on your own*. Any suspicion of collaboration will be flagged by me and in this case the default will be that you will receive 0 on the question. You can appeal later if you think I was wrong. If, upon further investigation, I still conclude that there was collaboration then I will pursue the consequences of academic dishonesty, which can ultimately mean withdrawal from the university, quite a bit more vigorously.

- The exam has 5 questions, each worth 15 marks. STAT 464 students must do question 1-4 for a total of 60 marks and STAT 864 students must do all 5 questions for a total of 75 marks. Part marks are shown to the right.

- The exam is a "take-home" 24 hour exam. The exam is open book. This means that you can use your notes, the textbook, and your computer.

- For each question, write your solution using your own paper. You may use a tablet if that is more convenient for you. Begin each solution at the start of a fresh page, and put your student number at the start of each solution.

- **The solutions are to be submitted through crowdmark.** You will receive an invitation from crowdmark shortly before the exam is emailed to you.

- **The deadline for submission of your solutions is 9am on April 27, Kingston time**.

Instructions continued on page 2.

- "The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
-
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. Marks per part question are shown in brackets at the right margin.

Student Number

1 (a) Let $X_t = s_t^{(2)} + s_t^{(3)} + s_t^{(4)}$, where $\{s_t^{(2)}\}$, $\{s_t^{(3)}\}$, and $\{s_t^{(4)}\}$, are seasonal components with periods 2, 3, and 4, respectively. Find a linear filter $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ (i.e., $a_0, \ldots, a_5$ are all nonzero and are the only nonzero coefficients of the filter) such that the output of the filter when applied to $\{X_t\}$ is a constant for all t. [8]

Solution: We use the fact that $s_t^{(2)} + s_{t-1}^{(2)}$, $s_t^{(3)} + s_{t-1}^{(3)} + s_{t-2}^{(3)}$, and $s_t^{(4)} + s_{t-1}^{(4)} + s_{t-2}^{(4)} + s_{t-3}^{(4)}$ are all constant for every t. Applying the filter to X_t gives

$$\begin{aligned}
& a_0 s_t^{(2)} + a_1 s_{t-1}^{(2)} + a_2 s_{t-2}^{(2)} + a_3 s_{t-3}^{(2)} + a_4 s_{t-4}^{(2)} + a_5 s_{t-5}^{(2)} \\
+ \; & a_0 s_t^{(3)} + a_1 s_{t-1}^{(3)} + a_2 s_{t-2}^{(3)} + a_3 s_{t-3}^{(3)} + a_4 s_{t-4}^{(3)} + a_5 s_{t-5}^{(3)} \\
+ \; & a_0 s_t^{(4)} + a_1 s_{t-1}^{(4)} + a_2 s_{t-2}^{(4)} + a_3 s_{t-3}^{(4)} + a_4 s_{t-4}^{(4)} + a_5 s_{t-5}^{(4)} \\
= \; & (a_0 + a_2 + a_4) s_t^{(2)} + (a_1 + a_3 + a_5) s_{t-1}^{(2)} \\
+ \; & (a_0 + a_3) s_t^{(3)} + (a_1 + a_4) s_{t-1}^{(3)} + (a_2 + a_5) s_{t-2}^{(3)} \\
+ \; & (a_0 + a_4) s_t^{(4)} + (a_1 + a_5) s_{t-1}^{(4)} + a_2 s_{t-2}^{(4)} + a_3 s_{t-3}^{(4)}.
\end{aligned}$$

Therefore, the output will be constant if

$$a_0 + a_2 + a_4 = a_1 + a_3 + a_5 \tag{1}$$

$$a_0 + a_3 = a_1 + a_4 = a_2 + a_5 \tag{2}$$

$$a_0 + a_4 = a_1 + a_5 = a_2 = a_3 \tag{3}$$

From Eq.(3) we have $a_2 = a_3$. Then Eq.(2) implies $a_0 = a_5$, and then Eq.(1) implies $a_1 = a_4$. Then in terms of a_0, a_1, a_2, the remaining equalities in Eqs.(2) and (3) reduce to $a_0 + a_2 = 2a_1$ and $a_0 + a_1 = a_2$. The first equation gives $a_2 = 2a_1 - a_0$. Plugging this into the second equation gives $a_0 + a_1 = 2a_1 - a_0$, or $a_1 = 2a_0$. Then $a_2 = 3a_0$. Thus, any solution of the form $\{a_0, 2a_0, 3a_0, 3a_0, 2a_0, a_0\}$ will satisfy Eqs.(1)-(3). Setting $a_0 = 1$, we can take our filter to be $\{1, 2, 3, 3, 2, 1\}$.

(b) Let d_1 and d_2 be positive integers, with $d_1 \neq d_2$. Let $\{s_t^{(d_1)}\}$ and $\{s_t^{(d_2)}\}$ be seasonal components with periods d_1 and d_2, respectively. Let $X_t = s_t^{(d_1)} + s_t^{(d_2)}$ and let $Y_t = \nabla_{d_1} X_t$. For each of the following statements, say whether it is **TRUE** or **FALSE**. If **TRUE** prove the statement, and if **FALSE** give a counterexample.

(i) $\nabla_{d_1 d_2} X_t$ is a constant. [2]

Solution: TRUE. We have

$$\begin{aligned} \nabla_{d_1 d_2} X_t &= (s_t^{(d_1)} + s_t^{(d_2)}) - (s_{t-d_1 d_2}^{(d_1)} + s_{t-d_1 d_2}^{(d_2)}) \\ &= (s_t^{(d_1)} - s_{t-d_1 d_2}^{(d_1)}) + (s_t^{(d_2)} - s_{t-d_1 d_2}^{(d_2)}). \end{aligned}$$

Both terms in parentheses in the last line above are equal to 0 for all t, and so $\nabla_{d_1 d_2} X_t = 0$ for all t.

(ii) $\nabla_{d_1 + d_2} X_t$ is a constant. [2]

Solution: FALSE. Take $d_1 = 2$ and $d_2 = 3$. Then $d_1 + d_2 = 5$ and $\nabla_{d_1+d_2} X_t = \nabla_5 X_t = s_t^{(2)} + s_t^{(3)} - s_{t-5}^{(2)} - s_{t-5}^{(3)} = (s_t^{(2)} - s_{t-1}^{(2)}) + (s_t^{(3)} - s_{t-2}^{(3)})$. Neither of the terms in parentheses need be constant for all t.

(iii) $\nabla_{d_2} Y_t$ is a constant. [3]

Solution: TRUE. We have

$$\begin{aligned} Y_t - Y_{t-d_2} &= (X_t - X_{t-d_1}) - (X_{t-d_2} - X_{t-d_2-d_1}) \\ &= (s_t^{(d_1)} + s_t^{(d_2)} - (s_{t-d_1}^{(d_1)} + s_{t-d_1}^{(d_2)})) - (s_{t-d_2}^{(d_1)} + s_{t-d_2}^{(d_2)} - (s_{t-d_2-d_1}^{(d_1)} + s_{t-d_2-d_1}^{(d_2)})) \\ &= (s_t^{(d_1)} - s_{t-d_1}^{(d_1)}) - (s_{t-d_2}^{(d_1)} - s_{t-d_2-d_1}^{(d_1)}) + (s_t^{(d_2)} - s_{t-d_2}^{(d_2)}) - (s_{t-d_1}^{(d_2)} - s_{t-d_1-d_2}^{(d_2)}). \end{aligned}$$

All terms in parentheses in the last line above are equal to zero for all t, and so $\nabla_{d_2} Y_t = 0$ for all t.

2 (a) Let $\{Y_t\}$ be a weakly stationary process and let $X_t = B^r Y_t$, where B is the backshift operator and r is a positive integer. Do $\{Y_t\}$ and $\{X_t\}$ have the same ACVF? If true, prove your answer. If false, give a counterexample. [2]

Solution: Yes, $\{X_t\}$ and $\{Y_t\}$ do have the same ACVF since $\gamma_X(h) = \text{Cov}(X_t, X_{t-h}) = \text{Cov}(Y_{t-r}, Y_{t-h-r}) = \gamma_Y(h)$.

(b) Let $\{Y_t\}$ be a weakly stationary process and let $X_t = \nabla_r Y_t$, where ∇_r is the lag r difference operator and r is a positive integer. Do $\{Y_t\}$ and $\{X_t\}$ have the same ACVF? If true, prove your answer. If false, give a counterexample. [3]

Solution: No, $\{X_t\}$ and $\{Y_t\}$ need not have the same ACVF. The ACVF of $\{X_t\}$ will be $\gamma_X(h) = \text{Cov}(X_t, X_{t-h}) = \text{Cov}(Y_t - Y_{t-r}, Y_{t-h} - Y_{t-h-r}) = 2\gamma_Y(r) - \gamma_Y(h+r) - \gamma_Y(h-r)$. For example, take $\{Y_t\}$ to be an MA(1) process, $h = 2$, and $r = 0$.

For parts (c) and (d) below, use the following notation and definitions. Let $\{Z_t\}$ be a zero-mean WN(σ^2) process and let $\{s_t\}$ be a seasonal component with period $d \geq 2$ (note that this implies that $s_t = c$ for all t, where c is a constant, is not allowed). Let $W_t = s_t Z_t$.

(c) Show that $\{W_t\}$ is not stationary. [2]

Solution: If we look at $\text{Var}(W_t) = s_t^2\sigma^2$ and $\text{Var}(W_{t+1}) = s_{t+1}^2\sigma^2$. If s_t^2 is not constant for all t then $\text{Var}(W_t)$ will depend on t and $\{W_t\}$ will not be stationary. Note: the question should have been "Show that $\{W_t\}$ is not necessarily stationary." If the solution just says the variance depends on t then full marks. If the solution says the statement is wrong (and says why) then also full marks.

(d) Suppose d is odd, say $d = 2q + 1$, where q is a positive integer. Also assume that $s_{d-1} \neq 0$. Let $X_t = \sum_{j=0}^{2q} s_j Z_{t-q+j}$. Show that $\{X_t\}$ is stationary and compute its ACVF and ACF. [8]

Solution: $\{X_t\}$ is a linear process (the output of a linear filter applied to a white noise process) and so is stationary. For the ACVF, one approach is to apply B^q to $\{X_t\}$ (which has the same ACVF by part(a)) and write $B^q X_t = \sum_{j=0}^{2q} s_j Z_{t-2q+j} = \sum_{j=0}^{2q} s_{2q-j} Z_{t-j}$. Written in this form we can apply Proposition 2.2.1 (Eq.2.2.5) to write

$$\gamma_X(h) = \sum_{j=h}^{2q} s_{2q-j} s_{2q-j+h} \sigma^2 = \sigma^2 \sum_{j=0}^{2q-h} s_j s_{j+h},$$

for $|h| \leq 2q$, and $\gamma_X(h) = 0$ for $|h| > 2q$. The ACF is then

$$\rho_X(h) = \frac{\sum_{j=0}^{2q-h} s_j s_{j+h}}{\sum_{j=0}^{2q} s_j^2} \quad \text{for } |h| \leq 2q \quad \text{and} \quad \rho_X(h) = 0 \text{ for } |h| > 2q.$$

3. Let $\{X_t\}$ be an MA(2) process with parameters θ_1, -1, and σ^2 (i.e., $X_t = Z_t + \theta_1 Z_{t-1} - Z_{t-2}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process).

(a) What are the approximate variances of the sample autocorrelations $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ based on $X_1, \ldots, X_n$ for n large? Express your answers in terms of θ_1 and simplify as much as possible. What are these approximate variances when $\theta_1 = 0$? [10]

Solution: We need to apply Bartlett's formula in Eq.(2.4.10) to compute w_{11} and w_{22}, in the case of the given MA(2) process, for which $\rho_X(0) = 1$, $\rho_X(1) = \rho_X(-1) = 0$, $\rho_X(2) = \rho_X(-2) = \frac{-1}{2+\theta_1^2}$, and $\rho_X(h) = 0$ for $|h| > 2$. Doing so, we have

$$\begin{aligned} w_{11} &= (\rho_X(2)+1)^2 + \rho_X(2)^2 = \left(\frac{1+\theta_1^2}{2+\theta_1^2}\right)^2 + \frac{1}{(2+\theta_1^2)^2} = \frac{2+2\theta_1^2+\theta_1^4}{(2+\theta_1^2)^2} \\ w_{22} &= (1-2\rho_X(2)^2)^2 + \rho_X(2)^2 = \left(1 - \frac{2}{(2+\theta_1^2)^2}\right)^2 + \frac{1}{(2+\theta_1^2)^2} \\ &= 1 - \frac{3}{(2+\theta_1^2)^2} + \frac{4}{(2+\theta_1^2)^4}. \end{aligned}$$

Then $\text{Var}(\hat{\rho}_X(1)) \approx \frac{1}{n} w_{11}$ and $\text{Var}(\hat{\rho}_X(2)) \approx \frac{1}{n} w_{22}$ for large n. If $\theta_1 = 0$ then these simplify to $\text{Var}(\hat{\rho}_X(1)) \approx \frac{1}{2n}$ and $\text{Var}(\hat{\rho}_X(2)) \approx \frac{1}{2n}$.

(b) What is the approximate covariance and correlation between the sample autocorrelations $\hat{\rho}_X(1)$ and $\hat{\rho}_X(2)$ based on $X_1, \ldots, X_n$ for n large? [3]

Solution: For large n, $\text{Cov}(\hat{\rho}_X(1), \hat{\rho}_X(2)) \approx \frac{1}{n} w_{12}$, where w_{12} is given by Bartlett's formula in Eq.(2.4.10). All terms in Eq.(2.4.10) for $i = 1$ and $j = 2$ are equal to 0 for the given $\rho_X(h)$ and so we have that $\text{Cov}(\hat{\rho}_X(1), \hat{\rho}_X(2)) \approx 0$.

(c) What is the approximate variance of $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ for large n? [2]

Solution: From Section 2.4.1, we have that the approximate variance of $\overline{X}_n$ for large n is $\frac{1}{n}\sum_{h=-\infty}^{\infty} \gamma_X(h)$. For the given MA(2) process we have $\gamma_X(0) = \sigma^2(2+\theta_1^2)$, $\gamma_X(1) = \gamma_X(-1) = 0$, $\gamma_X(2) = \gamma_X(-2) = -\sigma^2$, and $\gamma_X(h) = 0$ for $|h| > 2$. Thus,

$$\text{Var}(\overline{X}_n) \approx \frac{\sigma^2}{n}(2 + \theta_1^2 - 2) = \frac{\theta_1^2 \sigma^2}{n}.$$

4.(a) Let $\{X_t\}$ be a zero-mean stationary process. Let ρ_1, ρ_2, and ρ_3 denote the ACF of $\{X_t\}$ at lags 1, 2, and 3, respectively. Find $P(X_2 \mid X_1)$, $P(X_3 \mid X_2, X_1)$, and $P(X_2 \mid X_3, X_1)$, expressing the coefficients in these predictions in terms of ρ_1, ρ_2, and ρ_3. [7]

Solution: Letting $\gamma_X(h)$ denote the ACVF of $\{X_t\}$, the prediction $P(X_2 \mid X_1)$ is aX_1, where a is the solution to $\gamma_X(0)a = \gamma_X(1)$. This gives $a = \rho_1$. Thus, $P(X_2 \mid X_1) = \rho_1 X_1$. The prediction $P(X_3 \mid X_2, X_1)$ is $a^T(X_2, X_1)$, where $a = (a_1, a_2)^T$ is the solution to

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{1-\rho_1^2} \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \frac{1}{1-\rho_1^2} \begin{bmatrix} \rho_1(1-\rho_2) \\ \rho_2 - \rho_1^2 \end{bmatrix}.$$

Thus, $P(X_3 \mid X_2, X_1) = \frac{1}{1-\rho_1^2}(\rho_1(1-\rho_2)X_2 + (\rho_2 - \rho_1^2)X_1)$. For $P(X_2 \mid X_3, X_1)$, this prediction is $a^T(X_3, X_1)$, where $a = (a_1, a_2)^T$ is the solution to

$$\begin{bmatrix} \gamma_X(0) & \gamma_X(2) \\ \gamma_X(2) & \gamma_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(1) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_1 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{1-\rho_2^2} \begin{bmatrix} 1 & -\rho_2 \\ -\rho_2 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_1 \end{bmatrix} = \frac{1}{1-\rho_2^2} \begin{bmatrix} \rho_1(1-\rho_2) \\ \rho_1(1--\rho_2) \end{bmatrix}.$$

Thus, $P(X_2 \mid X_3, X_1) = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}(X_3 + X_1)$. (Note: the predictions do not depend on ρ_3). To summarize,

$$\begin{aligned} P(X_2 \mid X_1) &= \rho_1 X_1 \\ P(X_3 \mid X_2, X_1) &= \frac{1}{1-\rho_1^2}(\rho_1(1-\rho_2)X_2 + (\rho_2 - \rho_1^2)X_1) \\ P(X_2 \mid X_3, X_1) &= \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}(X_3 + X_1) \end{aligned}$$

(b) Let $\{X_t\}$ be an MA(2) process with parameters θ_1, -1, and σ^2 (i.e., $X_t = Z_t + \theta_1 Z_{t-1} - Z_{t-2}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process). Find $P(X_4 \mid X_3, X_2, X_1)$, $P(X_4 \mid X_2, X_1)$, and $P(X_4 \mid X_1)$ and the mean squared errors of each of these predictions in terms of the parameters θ_1 and σ^2. [8]

Solution: The ACF for this process (cf. Problem 3(a)) is $\rho_X(0) =$, $\rho_X(1) = \rho_X(-1) = 0$, $\rho_X(2) = \rho_X(-2) = \frac{-1}{2+\theta_1^2}$, and $\rho_X(h) = 0$ for $|h| > 2$. The prediction $P(X_4 \mid X_3, X_2, X_1)$ is $a^T(X_3, X_2, X_1)$, where $a = (a_1, a_2, a_3)^T$ is the solution to

$$\begin{bmatrix} 1 & 0 & \rho_X(2) \\ 0 & 1 & 0 \\ \rho_X(2) & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \rho_X(2) \\ 0 \end{bmatrix},$$

These equations give $a_2 = \rho_X(2)$ and $a_1 = a_3 = 0$. Thus, $P(X_4 \mid X_3, X_2, X_1) = \frac{-1}{2+\theta_1^2} X_2$. The MSE of this prediction is $\gamma_X(0) - a^T(0, \gamma_X(2), 0) = \sigma^2(2 + \theta_1^2 + \frac{1}{2+\theta_1^2})$ (see Problem 3(c) for the ACVF of the given MA(2) process). Alternatively, one can note that X_4 is uncorrelated with both X_3 and X_1, and so $P(X_4 \mid X_3, X_2, X_1) = P(X_4 \mid X_2)$ (which is $\rho_X(2)X_2$). For $P(X_4 \mid X_2, X_1)$, this prediction is given by $a^T(X_2, X_1)$, where $a = (a_1, a_2)^T$ is the solution to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \rho_X(2) \\ 0 \end{bmatrix},$$

the solution of which is $a_1 = \rho_X(2)$ and $a_2 = 0$. Thus, $P(X_4 \mid X_2, X_1) = -\frac{1}{2+\theta_1^2} X_2$. The MSE of this prediction is $\gamma_X(0) - a^T(\gamma_X(2), 0) = \sigma^2(2 + \theta_1^2 + \frac{1}{2+\theta_1^2})$. Alternatively, one can note that X_4 is uncorrelated with X_1 and so $P(X_4 \mid X_2, X_1) = P(X_4 \mid X_2)$. For $P(X_4 \mid X_1)$ we can simply observe that X_4 is uncorrelated with X_1 and so $P(X_4 \mid X_1) = E[X_4] = 0$. The MSE of this prediction is $\gamma_X(0) = \sigma^2(2 + \theta_1^2)$. To summarize,

$$\begin{aligned} P(X_4 \mid X_3, X_2, X_1) &= -\frac{1}{2+\theta_1^2} X_2 \text{ with MSE } \sigma^2 \left(2 + \theta_1^2 + \frac{1}{2+\theta_1^2}\right) \\ P(X_4 \mid X_2, X_1) &= -\frac{1}{2+\theta_1^2} X_2 \text{ with MSE } \sigma^2 \left(2 + \theta_1^2 + \frac{1}{2+\theta_1^2}\right) \\ P(X_4 \mid X_1) &= 0 \text{ with MSE } \sigma^2(2 + \theta_1^2). \end{aligned}$$

***5.** Let $X_t = \phi X_{t-1} + \sum_{j=0}^{q} \theta^j Z_{t-j}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, $|\phi| < 1$, $|\theta| < 1$, $\theta \neq \phi$, $\phi \neq 0$, $\theta \neq 0$, and $q \geq 1$.

(a) Find the coefficients ψ_k in the representation $X_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}$. You must show your work in deriving the ψ_k. The ψ_k must be expressed in terms of ϕ and θ only and be as simplified as possible (there should not be any sums in your expression for ψ_k). [8]

Solution: In terms of the backward shift operator B the equations defining $\{X_t\}$ can be written as $(1 - \phi B)X_t = (\sum_{j=0}^{q} (\theta B)^j) Z_t$, from which we can write

$$X_t = \Big((1 - \phi B)^{-1} \sum_{j=0}^{q} (\theta B)^j \Big) Z_t = \Big(\sum_{i=0}^{\infty} (\phi B)^i \sum_{j=0}^{q} (\theta B)^j \Big) Z_t.$$

From the above we can see that $\psi_k = 0$ for $k < 0$ while for $k \geq 0$, ψ_k is the coefficient of B^k when we multiply out $(\sum_{i=0}^{\infty} (\phi B)^i \sum_{j=0}^{q} (\theta B)^j)$. We have

$$\begin{aligned}
\sum_{i=0}^{\infty} (\phi B)^i \sum_{j=0}^{q} (\theta B)^j &= \sum_{i=0}^{\infty} \sum_{j=0}^{q} \phi^i \theta^j B^{i+j} \quad \text{(set } k = i + j \text{ then } i = k - j) \\
&= \sum_{k=0}^{q} \sum_{j=0}^{k} \phi^{k-j} \theta^j B^k + \sum_{k=q+1}^{\infty} \sum_{j=0}^{q} \phi^{k-j} \theta^j B^k \quad \text{(careful on limits!)} \\
&= \sum_{k=0}^{q} \left[\phi^k \sum_{j=0}^{k} \left(\frac{\theta}{\phi} \right)^j \right] B^k + \sum_{k=q+1}^{\infty} \left[\phi^k \sum_{j=0}^{q} \left(\frac{\theta}{\phi} \right)^j \right] B^k.
\end{aligned}$$

Thus, we see that

$$\psi_k = \begin{cases} \phi^k \sum_{j=0}^{k} \left(\frac{\theta}{\phi} \right)^j = \phi^k \frac{1 - (\theta/\phi)^{k+1}}{1 - (\theta/\phi)} & k = 0, \ldots, q \\ \\ \phi^k \sum_{j=0}^{q} \left(\frac{\theta}{\phi} \right)^j = \phi^k \frac{1 - (\theta/\phi)^{q+1}}{1 - (\theta/\phi)} & k > q \end{cases}.$$

For $k > q$ note that $\psi_k = \phi^{k-q} \psi_q$.

(b) For $q = 2$ find the ACVF, $\gamma_X(h)$, of $\{X_t\}$. [7]

Solution: