STUDENT NUMBER:

STAT 464/864 — Midterm

Solutions, 2022

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Instructions: Total points = 20 for 464 and 30 for 864. Duration = 50 minutes.

There are **3 questions**, each worth 10 marks. **STAT 864** must answer **all questions**, writing clearly in the space provided. **STAT 464** students must answer **questions 1 and 2**, writing clearly in the space provided. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.

The exam is closed book. A single 8.5 by 11 inch sheet of notes, written on both sides is allowed. Also, a simple, non-programmable calculator (Casio 911) is allowed. Good luck!

1	**2**	**3**	Total (464)	Total (864)
/10	/10	/10	/20	/30

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Space for additional work. Indicate clearly which question you are continuing if you use this page.

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1. Let $Y_t = c_0 + c_1 t + s_t + Z_t$, where c_0 and c_1 are constants, $\{s_t\}$ is a seasonal component with period d, and $\{Z_t\}$ is a zero-mean $\mathrm{WN}(\sigma^2)$ process. Give a linear filter with exactly 2 nonzero filter coefficients such that when applied to the input $\{Y_t\}$ the output is stationary, and compute the ACF of the output process. [10]

Solution: Consider ∇_d, the lag d difference operator, which corresponds to the linear filter $\{a_j\}$ with $a_0 = 1$, $a_d = -1$, and $a_j = 0$ otherwise. Applying this filter to $c_0 + c_1 t$ gives the output $\nabla_d(c_0 + c_1 t) = c_0 + c_1 t - (c_0 + c_1(t-d)) = c_1 d$, which is a constant, and so the linear trend is eliminated. Applied to s_t we get $\nabla_d s_t = s_t - s_{t-d} = 0$ since $\{s_t\}$ has period d. Therefore, applying the filter to $\{Y_t\}$ gives the output $X_t = \nabla_d Y_t = c_1 d + Z_t - Z_{t-d}$. Then $\mathrm{Var}(X_t) = \mathrm{Var}(Z_t) + \mathrm{Var}(Z_{t-d}) = 2\sigma^2$, which doesn't depend on t. Also, for $h \neq 0$, $\mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(Z_t - Z_{t-d}, Z_{t+h} - Z_{t+h-d}) = -\sigma^2$ if $h = \pm d$, and is equal to 0 otherwise. Again, this does not depend on t. So we can conclude that $\{X_t\}$ is stationary with ACF given by $\rho_X(0) = 1$, $\rho_X(\pm d) = -\frac{1}{2}$, and $\rho_X(h) = 0$ for $h \neq 0, \pm d$.

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2. *Sum and product of MA(1) processes.* Let $U_t = W_t + \theta_1 W_{t-1}$ and $V_t = Z_t + \theta_2 Z_{t-1}$, where θ_1 and θ_2 are nonzero, $\{W_t\}$ and $\{Z_t\}$ are zero-mean WN(σ^2) processes. Assume that $\{W_t\}$ and $\{Z_t\}$ are independent; that is W_s and Z_t are independent for all s, t. Let $X_t = U_t + V_t$ and $Y_t = U_t V_t$.

(a) Show that $\{X_t\}$ and $\{Y_t\}$ are stationary processes and compute $\rho_X(h)$ and $\rho_Y(h)$, the ACFs of $\{X_t\}$ and $\{Y_t\}$, respectively. [7]

(b) If $\theta_2 = -\theta_1$ show that $\{X_t\}$ is a white noise process and compute the variance of this white noise process. [3]

Solution: Let $\gamma_U(h)$ denote the ACVF of $\{U_t\}$ and let $\gamma_V(h)$ denote the ACVF of $\{V_t\}$. Since $\{U_t\}$ and $\{V_t\}$ are MA(1) processes, we know that $\gamma_U(0) = \sigma^2(1+\theta_1^2)$, $\gamma_U(\pm 1) = \sigma^2\theta_1$, and $\gamma_U(h) = 0$ for $|h| > 1$. Similarly, $\gamma_V(0) = \sigma^2(1+\theta_2^2)$, $\gamma_V(\pm 1) = \sigma^2\theta_2$, and $\gamma_V(h) = 0$ for $|h| > 1$.

(a) Starting with $\{X_t\}$, we have $\text{Cov}(X_t, X_{t+h}) = \text{Cov}(U_t+V_t, U_{t+h}+V_{t+h}) = \text{Cov}(U_t, U_{t+h})+\text{Cov}(V_t, V_{t+h}) = \gamma_U(h) + \gamma_V(h)$. This does not depend on t so $\{X_t\}$ is stationary. Let $\gamma_X(h)$ denote the ACVF of $\{X_t\}$. We have $\gamma_X(0) = \sigma^2(2+\theta_1^2+\theta_2^2)$, $\gamma_X(\pm 1) = \sigma^2(\theta_1+\theta_2)$, and $\gamma_X(h) = 0$ for $|h| > 1$. Then the ACF of $\{X_t\}$ is $\rho_X(0) = 1$, $\rho_X(\pm 1) = \frac{\theta_1+\theta_2}{2+\theta_1^2+\theta_2^2}$, and $\rho_X(h) = 0$ for $|h| > 1$. Turning to $\{Y_t\}$, we have $\text{Cov}(Y_t, Y_{t+h}) = E[Y_t Y_{t+h}] = E[U_t V_t U_{t+h} V_{t+h}] = E[U_t U_{t+h}]E[V_t V_{t+h}] = \gamma_U(h)\gamma_V(h)$, where the second last equality is true because $\{U_t\}$ and $\{V_t\}$ are independent processes. This does not depend on t so $\{Y_t\}$ is stationary. Let $\gamma_Y(h)$ denote the ACVF of $\{Y_t\}$. We have $\gamma_Y(0) = \sigma^4(1+\theta_1^2)(1+\theta_2^2)$, $\gamma_Y(\pm 1) = \sigma^4\theta_1\theta_2$, and $\gamma_Y(h) = 0$ for $|h| > 1$. Then the ACF of $\{Y_t\}$ is $\rho_Y(0) = 1$, $\rho_Y(\pm 1) = \frac{\theta_1\theta_2}{(1+\theta_1^2)(1+\theta_2^2)}$, and $\rho_Y(h) = 0$ for $|h| > 1$.

(b) If $\theta_2 = -\theta_1$ then $\gamma_X(0) = \sigma^2(2+2\theta_1^2)$ and $\gamma_X(\pm 1) = 0$ from part(a). Therefore, in this case $\{X_t\}$ is a white noise process with variance $2\sigma^2(1+\theta_1^2)$.

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3.(a) Let $\{X_t\}$ be an MA(1) process with MA coefficient $\theta \in \mathbb{R}$, $\theta \neq 0$; i.e., $X_t = Z_t + \theta Z_{t-1}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process. Compute $\rho(1)$, the autocorrelation at lag 1, and show that $|\rho(1)| \leq 1/2$ for all θ. [7]

(b) Let r be such that $r \in (0.75, 1)$. Show that there does not exist any stationary process $\{X_t\}$ such that $\rho_X(1) = r$ and $\rho_X(2) = 0$, where $\rho_X(\cdot)$ is the ACF of $\{X_t\}$. *Hint:* Consider the vector $(1, -1, 1)^T$ to show that the correlation matrix of $(X_1, X_2, X_3)^T$ is not nonnegative definite. [3]

Solution:

(a) For the MA(1) process with coefficient θ, the ACF at lag 1 is $\rho(1) = \frac{\theta}{1+\theta^2}$. Note that this is an odd function of θ, so to show that $|\frac{\theta}{1+\theta^2}| \leq \frac{1}{2}$ it suffices to show that $\frac{\theta}{1+\theta^2} \leq \frac{1}{2}$ for $\theta \geq 0$. This is equivalent to $2\theta \leq 1 + \theta^2$, or $\theta^2 - 2\theta + 1 \geq 0$, or $(\theta - 1)^2 \geq 0$, which is true.

(b) If $\{X_t\}$ is stationary with $\rho_X(1) = r$ and $\rho_X(2) = 0$ then the correlation matrix of $(X_1, X_2, X_3)^T$ is

$$R_3 = \begin{bmatrix} 1 & r & 0 \\ r & 1 & r \\ 0 & r & 1 \end{bmatrix}.$$

With the vector $a = (1, -1, 1)^T$ one can easily compute $a^T R_3 a = (1)(1-r) + (-1)(2r-1) + (1)(1-r) = 3 - 4r$. If $r \in (0.75, 1)$ then $a^T R_3 a$ will be negative. This is impossible for a correlation matrix from a stationary process, which must be nonnegative definite. Based on this contradiction, $\{X_t\}$ cannot be stationary.

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