STAT 464/864

Midterm Practice Problems – Solutions, 2022

1. Suppose $\{s_t\}$ is a seasonal component with period $d = 2$. Show that $|\nabla^2 s_t|$ has no seasonal component, where ∇ is the difference operator.

Solution: $\nabla^2 s_t = s_t - 2s_{t-1} + s_{t-2} = s_t - s_{t-1} - (s_{t-1} - s_{t-2})$. But if $\{s_t\}$ has period 2 then $s_t - s_{t-1} = -(s_{t-1} - s_{t-2})$ and so $\nabla^2 s_t = 2(s_t - s_{t-1})$, and $|\nabla^2 s_t| = 2|s_t - s_{t-1}|$. This is the same for every t if $\{s_t\}$ has period 2, and so has no seasonal component.

2. Let $\{s_t\}$ and $\{r_t\}$ be seasonal components with periods a and b, respectively, where a and b are distinct positive integers that do not share any prime factors. Let $X_t = s_t + r_t + Z_t$, where $\{Z_t\}$ is a zero mean $\text{WN}(\sigma^2)$ process. Find d_1 and d_2 such that $\nabla_{d_1}\nabla_{d_2}X_t$ is stationary, where ∇_d is the lag d difference operator, and find the ACF of $\nabla_{d_1}\nabla_{d_2}X_t$.

Solution: Take $d_1 = a$ and $d_2 = b$. Then $\nabla_a\nabla_b s_t = s_t - s_{t-b} - s_{t-a} + s_{t-(a+b)}$. But $s_t - s_{t-a} = 0$ and $s_{t-b} - s_{t-b-a} = 0$. Similarly, $\nabla_a\nabla_b r_t = r_t - r_{t-b} - r_{t-a} + r_{t-(a+b)}$. But $r_t - r_{t-b} = 0$ and $r_{t-a} - r_{t-a-b} = 0$. So $\nabla_a\nabla_b(s_t + r_t) = 0$ and $\nabla_a\nabla_b X_t = \nabla_a\nabla_b Z_t = Z_t - Z_{t-a} - Z_{t-b} + Z_{t-(a+b)}$. Then $\nabla_a\nabla_b Z_t$ is the output of a linear filter $\{\psi_j\}$ with $\psi_0 = 1$, $\psi_a = -1$, $\psi_b = -1$, $\psi_{a+b} = 1$, and $\psi_j = 0$ for $j \notin \{0, a, b, a+b\}$. $\{\nabla_a\nabla_b X_t\}$ is thus a linear process, and hence is stationary. It is also causal and has ACVF $\gamma_X(h) = \sum_{j=0}^{\infty}\psi_j\psi_{j+h}$. For $h \ge 0$, the only h for which $\gamma_X(h)$ is nonzero are $h = 0, a, b, |b-a|$, and $a+b$. For these h we have $\gamma_X(0) = 4\sigma^2$, $\gamma_X(a) = -2\sigma^2$, $\gamma_X(b) = -2\sigma^2$, $\gamma_X(|b-a|) = \sigma^2$, and $\gamma_X(a+b) = \sigma^2$. Finally, $\gamma_X(-h) = \gamma_X(h)$. The ACF of $\{X_t\}$ is then given by $\rho_X(0) = 1$, $\rho_X(\pm a) = \rho_X(\pm b) = -\frac{1}{2}$, $\rho_X(\pm|b-a|) = \frac{1}{4}$, $\rho_X(\pm(a+b)) = \frac{1}{4}$, and $\rho_X(h) = 0$ otherwise.

3. Suppose $\{X_t\}$ and $\{Y_t\}$ are independent (i.e., X_r and Y_s are independent for every r and s), 0-mean, stationary processes with autocovariance functions $\gamma_X(h)$ and $\gamma_Y(h)$, respectively. Let $Z_t = X_tY_t$. What is the ACVF and ACF of $\{Z_t\}$?

Solution: Since $E[Z_t] = E[X_tY_t] = E[X_t]E[Y_t] = 0$ for all t the ACVF of $\{Z_t\}$ at lag h is $\gamma_Z(h) = E[Z_tZ_{t+h}] = E[X_tY_tX_{t+h}Y_{t+h}] = E[X_tX_{t+h}]E[Y_tY_{t+h}] = \gamma_X(h)\gamma_Y(h)$. Then the ACF of $\{Z_t\}$ is $\rho_Z(h) = \rho_X(h)\rho_Y(h)$, where $\rho_X(h)$ and $\rho_Y(h)$ are the ACFs of $\{X_t\}$ and $\{Y_t\}$, respectively, at lag h.

4. For d odd, give a symmetric linear filter (i.e., $a_{-j} = a_j$) that eliminates seasonal components with period d and passes linear trends. What if d is even?

Solution: For d odd set $a_j = \frac{1}{d}$ for $j = -\frac{d-1}{2}, \ldots, \frac{d-1}{2}$, and $a_j = 0$ otherwise. For d even set $a_j = \frac{1}{2d}$ for $j = -d, \ldots, -1$ and $j = 1, \ldots, d$, and set $a_j = 0$ otherwise.

5. Consider the exponential smoother with smoothing parameter $\alpha \in (0,1)$ and let $\{X_t\}$ be the result of applying this smoother to $\{Z_t\}$, where $\{Z_t\}$ is a zero-mean WN(σ^2) process, i.e., $X_t = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i Z_{t-i}$. Find the autocovariance function (ACVF) of $\{X_t\}$ at all lags h.

Solution: $\{X_t\}$ is the output of the linear filter $\{\psi_j\}$, where, $\psi_j = \alpha(1-\alpha)^j$ for $j \geq 0$ and $\psi_j = 0$ for $j < 0$. For $h \geq 0$, the ACVF of $\{X_t\}$ is

$$\gamma_X(h) = \sum_{j=0}^{\infty} \alpha(1-\alpha)^j \alpha(1-\alpha)^{j+h} = \alpha^2(1-\alpha)^h \frac{1}{1-(1-\alpha)^2} = (1-\alpha)^h \frac{\alpha}{2-\alpha},$$

and $\gamma_X(-h) = \gamma_X(h)$.

6. Let $X_t = Y_0 s_t + Y_1 s_{t-1}$, where $\{s_t\}$ is seasonal with period 2, and $Y_t = Z_t + \theta Z_{t-1}$ is an MA(1) process with MA coefficient θ, and $\{Z_t\}$ is a zero-mean WN(σ^2) process. Compute the ACF of $\{X_t\}$.

Solution: $\{X_t\}$ is clearly zero-mean. So for $\mathrm{Cov}(X_t, X_{t+h}) = E[X_t, X_{t+h}] = E[(Y_0 s_t + Y_1 s_{t-1})(Y_0 s_{t+h} + Y_1 s_{t+h-1})] = s_t s_{t+h} E[Y_0^2] + s_{t-1} s_{t+h-1} E[Y_1^2] + (s_t s_{t+h-1} + s_{t-1} s_{t+h}) E[Y_0 Y_1] = \gamma_Y(0)(s_t s_{t+h} + s_{t-1} s_{t+h-1}) + \gamma_Y(1)(s_t s_{t+h-1} + s_{t-1} s_{t+h})$, where $\gamma_Y(h)$ is the ACVF of $\{Y_t\}$. If h is even then $s_t s_{t+h} + s_{t-1} s_{t+h-1} = s_0^2 + s_1^2$ and $s_t s_{t+h-1} + s_{t-1} s_{t+h} = 2 s_0 s_1$ for every t, while if h is odd then $s_t s_{t+h} + s_{t-1} s_{t+h-1} = 2 s_0 s_1$ and $s_t s_{t+h-1} + s_{t-1} s_{t+h} = s_0^2 + s_1^2$ for every t. Therefore, $\mathrm{Cov}(X_t, X_{t+h})$ does not depend on t and so $\{X_t\}$ is stationary. For the MA(1) process with parameter θ, $\gamma_Y(0) = \sigma^2(1+\theta^2)$ and $\gamma_Y(1) = \sigma^2\theta$. For h even the ACVF of $\{X_t\}$ is then $\gamma_X(h) = \sigma^2(1+\theta^2)(s_0^2 + s_1^2) + 2\sigma^2\theta s_1 s_2$ and for h odd $\gamma_X(h) = 2\sigma^2(1+\theta^2) s_0 s_1 + \sigma^2\theta(s_0^2 + s_1^2)$. Then the ACF of $\{X_t\}$ is $\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = 1$ if h is even and equals $\frac{2(1+\theta^2)s_0 s_1 + \theta(s_0^2+s_1^2)}{(1+\theta^2)(s_0^2+s_1^2)+2\theta s_1 s_2}$ if h is odd.

7. Show that the two MA(1) processes

$$\begin{aligned} X_t &= Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2) \\ Y_t &= \tilde{Z}_t + \frac{1}{\theta}\tilde{Z}_{t-1}, \quad \{\tilde{Z}_t\} \sim \mathrm{WN}(0, \sigma^2\theta^2), \end{aligned}$$

where $0 < |\theta| < 1$, have the same autocovariance function.

Solution: The ACVF of $\{X_t\}$ is $\gamma_X(0) = \sigma^2(1+\theta^2)$, $\gamma_X(\pm 1) = \sigma^2\theta$, and $\gamma_X(h) = 0$ otherwise. The ACVF of $\{Y_t\}$ is $\gamma_Y(0) = \sigma^2\theta^2(1 + 1/\theta^2) = \sigma^2(\theta^2 + 1)$, $\gamma_Y(\pm 1) = \sigma^2\theta^2(1/\theta) = \sigma^2\theta$, and $\gamma_Y(h) = 0$ otherwise. Therefore, $\{X_t\}$ and $\{Y_t\}$ have the same ACVF.