Queen's University
Department of Mathematics and Statistics

STAT 464/864

Midterm Exam Solutions, 2021

- **THE SOLUTIONS ARE TO BE UPLOADED TO CROWDMARK**. You should have received an invitation from Crowdmark to submit your solutions.
- Write your solutions using your own paper or directly on the exam sheets (assuming you print out the exam). You may also use a writing tablet such as an ipad to write your solutions. If writing on paper, use a dark lead pencil or pen to write your solutions. Solutions which cannot be read because they are too faint will not be marked.
- Start each solution on a new page. Write your student number at the start of each solution and number each solution.
- **Duration = 90 minutes + 30 minutes to prepare your solutions for upload to Crowdmark and to submit your solutions to Crowdmark.**
- The exam is **open book**. This means you can use the class notes and the textbook. You may also use your computer. However, **you may not use any device for communication with another person (see the next item)**.
- **ABSOLUTELY ZERO COLLABORATION IS ALLOWED**! There is to be no collaboration in any form on any question on any part of the exam. All work on the exam must be completed *on your own*.
- The exam has 3 questions. Questions 1 and 2 are for STAT 464 students. STAT 864 students must do all 3 questions. Note that STAT 464 students *must* do questions 1 and 2. If you are in STAT 464 and do question 3 *it will not be marked*.
- Each question is worth 10 marks, for a total of 20 for STAT 464 and 30 for STAT 864. Marks per part question are shown in brackets at the right.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
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1. (a) Let $\{s_t\}$ and $\{r_t\}$ be seasonal components with periods a and b, respectively, where a and b are distinct positive integers that do not share any prime factors. Let $X_t = s_t r_t + Y_t$, where $Y_t = \phi Y_{t-1} + Z_t$ is an AR(1) process with AR coefficient ϕ satisfying $|\phi| < 1$ and $\{Z_t\}$ is a zero mean WN(σ^2) process. Find d such that the lag d difference operator applied to $\{X_t\}$ (i.e., $\nabla_d X_t$) is stationary, and find the ACF of $\nabla_d X_t$. [6]

Solution: Take $d = ab$. Then $\nabla_{ab} s_t r_t = s_t r_t - s_{t-ab} r_{t-ab}$. But $s_{t-ab} = s_t$ and $r_{t-ab} = r_t$ so $\nabla_{ab} s_t r_t = 0$. Then applying ∇_{ab} to X_t gives $\nabla_{ab} X_t = Y_t - Y_{t-ab}$. Let $\gamma(h)$ denote the ACVF of $\{\nabla_{ab} X_t\}$ and $\gamma_Y(h)$ denote the ACVF of $\{Y_t\}$. Then for $h \geq 0$,

$$\gamma(h) = \text{Cov}(Y_t - Y_{t-ab}, Y_{t-h} - Y_{t-ab-h}) = 2\gamma_Y(h) - \gamma_Y(h + ab) - \gamma_Y(h - ab).$$

The ACVF of the given AR(1) process is $\gamma_Y(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2}$, which is valid for all lags h. For $\gamma(h)$ at $h = 0$ we have $\gamma(0) = 2(\gamma_Y(0) - \gamma_Y(ab)) = \frac{2\sigma^2(1-\phi^{ab})}{1-\phi^2}$. For $h > 0$ we have $\gamma(h) = \frac{\sigma^2}{1-\phi^2}(2\phi^h - \phi^{h+ab} - \phi^{|h-ab|})$. Then the ACF of $\{\nabla_{ab} X_t\}$ is

$$\rho(h) = \frac{2\phi^{|h|} - \phi^{|h+ab|} - \phi^{|h-ab|}}{2(1 - \phi^{ab})},$$

which is valid for all h.

(b) Let $\{s_t\}$ be a seasonal component with period d, and let a, b, and c be constants. Let $X_t = (a + bt + ct^2)s_t + Y_t$, where $\{Y_t\}$ is a zero mean stationary process. Show that $\nabla_d^3 X_t$ is stationary. [4]

Solution: Writing $\nabla_d^3 X_t$, we have, using the fact that $s_{t-d} = s_t$,

$$
\begin{aligned}
\nabla_d^3 X_t &= \nabla_d^2 \Big[(a + bt + ct^2)s_t + Y_t - \Big(a + b(t-d) + c(t-d)^2)s_{t-d} + Y_{t-d}\Big)\Big] \\
&= \nabla_d^2 \Big[((bd - cd^2) + 2cdt)s_t + Y_t - Y_{t-d}\Big] \\
&= \nabla_d \Big[((bd - cd^2) + 2cdt)s_t + Y_t - Y_{t-d} \\
&\qquad - \Big(((bd - cd^2) + 2cd(t-d))s_{t-d} + Y_{t-d} - Y_{t-2d}\Big)\Big] \\
&= \nabla_d \Big[2cd^2 s_t + Y_t - 2Y_{t-d} + Y_{t-2d}\Big] \\
&= 2cd^2 s_t + Y_t - 2Y_{t-d} + Y_{t-2d} - \Big(2cd^2 s_{t-d} + Y_{t-d} - 2Y_{t-2d} + Y_{t-3d}\Big) \\
&= Y_t - 3Y_{t-d} + 3Y_{t-2d} - Y_{t-3d}.
\end{aligned}
$$

From the above we see that $\{\nabla_d^3 X_t\}$ is the output of a linear filter applied to $\{Y_t\}$. Since the input $\{Y_t\}$ is a zero-mean stationary process, by Proposition 2.2.1 so is the output $\{\nabla_d^3 X_t\}$.

2. Let $\{Z_t\}$ be a zero mean zero WN(σ^2) process. For each of the following parts, state whether the given process $\{X_t\}$ is stationary or not. If it is not stationary, prove it. If it is stationary, compute its ACF.

(a) $X_t = (-1)^t Z_0 + (-1)^{t+1} Z_1$. [3]

Solution: We compute

$$\begin{aligned} \text{Cov}(X_t, X_{t+h}) &= \text{Cov}((-1)^t Z_0 + (-1)^{t+1} Z_1, (-1)^{t+h} Z_0 + (-1)^{t+h+1} Z_1) \\ &= (-1)^{2t+h}\sigma^2 + (-1)^{2t+h+2}\sigma^2 = (-1)^{2t+h}\sigma^2(1+1) = (-1)^h 2\sigma^2. \end{aligned}$$

We see that this does not depend on t for any lag h. Therefore, $\{X_t\}$ is stationary (it is obviously zero mean). The ACF of $\{X_t\}$ is then given by $\rho_X(h) = (-1)^h$.

(b) $X_t = Z_0 s_t + Z_1 s_{t-1}$, where $\{s_t\}$ is seasonal with period 2. [4]

Solution: We compute

$$\begin{aligned} \text{Cov}(X_t, X_{t+h}) &= \text{Cov}(Z_0 s_t + Z_1 s_{t-1}, Z_0 s_{t+h} + Z_1 s_{t-1+h}) \\ &= s_t s_{t+h}\sigma^2 + s_{t-1}s_{t-1+h}\sigma^2 \\ &= \begin{cases} (s_t^2 + s_{t-1}^2)\sigma^2 & \text{if } h \text{ is even} \\ 2 s_t s_{t-1}\sigma^2 & \text{if } h \text{ is odd} \end{cases}. \end{aligned}$$

But both $s_t^2 + s_{t-1}^2$ and $s_t s_{t-1}$ are the same for any t since $\{s_t\}$ has period 2. Therefore, $\text{Cov}(X_t, X_{t+h})$ does not depend on t and so the process $\{X_t\}$ is stationary (it is clearly zero mean). The ACF of $\{X_t\}$ is then given by $\rho_X(h) = 1$ if h is even and $\rho_X(h) = \frac{2s_0 s_1}{s_0^2 + s_1^2}$ if h is odd.

(c) $X_t = \sum_{j=0}^{d} \frac{1}{d+1} Z_{t-j}$, where d is a fixed positive integer. [3]

Solution: $\{X_t\}$ is a linear process with coefficients $\psi_j = \frac{1}{d+1}$ for $j = 0, \ldots, d$ and $\psi_j = 0$ otherwise. Since $\{X_t\}$ is a linear process it is stationary. By Proposition 2.2.1, the ACVF of $\{X_t\}$ is $\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2 = \sigma^2 \frac{1}{d+1} \sum_{j=0}^{d} \psi_{j+h}$. For $h > d$ the sum is 0. For $h = 0, \ldots, d$ the sum is $\frac{d+1-h}{d+1}$. Therefore, the ACVF of $\{X_t\}$ is

$$\gamma_X(h) = \sigma^2 \frac{d+1-|h|}{(d+1)^2} \quad \text{for } |h| \leq d$$

and $\gamma_X(h) = 0$ for $|h| > d$. Then the ACF of $\{X_t\}$ is given by

$$\rho_X(h) = \frac{d+1-|h|}{d+1} \quad \text{for } |h| \leq d$$

and $\rho_X(h) = 0$ for $|h| > d$.

***3.** Let $\{Z_t\}$ be a zero mean WN(σ^2) process, and let $X_t = Z_t + \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j-1} Z_{t-j}$. Find the ACVF and the ACF of $\{X_t\}$. [10]

Solution: $\{X_t\}$ is a linear process with coefficients $\psi_0 = 1$, $\psi_j = \left(\frac{1}{2}\right)^{j-1}$ for $j \geq 1$, and $\psi_j = 0$ for $j < 0$. By Proposition 2.2.1, the ACVF of $\{X_t\}$ is $\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$. Since $\psi_j = 0$ for $j < 0$, we have for $h > 0$

$$\begin{aligned}
\gamma_X(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \\
&= \sigma^2 \left(\frac{1}{2}\right)^{h-1} + \sigma^2 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j-1} \left(\frac{1}{2}\right)^{j+h-1} \\
&= \sigma^2 \left(\frac{1}{2}\right)^{h-1} + \sigma^2 \left(\frac{1}{2}\right)^{h} \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^{j-1} \\
&= \sigma^2 \left(\frac{1}{2}\right)^{h-1} \left[1 + \frac{1}{2}\frac{1}{1-1/4}\right] \\
&= \sigma^2 \left(\frac{1}{2}\right)^{h-1} \left(\frac{5}{3}\right).
\end{aligned}$$

For $h = 0$ we have $\gamma_X(0) = \sigma^2 \left(1 + \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^{j-1}\right) = \frac{7}{3}\sigma^2$. Then we have that the ACVF of $\{X_t\}$ is

$$\gamma_X(h) = \begin{cases} \frac{7}{3}\sigma^2 & \text{for } h = 0 \\ \left(\frac{5}{3}\right)\left(\frac{1}{2}\right)^{|h|-1} \sigma^2 & \text{for } h \neq 0 \end{cases}$$

and the ACF of $\{X_t\}$ is

$$\rho_X(h) = \begin{cases} 1 & \text{for } h = 0 \\ \left(\frac{5}{7}\right)\left(\frac{1}{2}\right)^{|h|-1} & \text{for } h \neq 0 \end{cases}.$$