1. In the following problems, first verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Write out the general solution of the system. Then find a particular solution that satisfies the given initial conditions.

(a) \( \mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \mathbf{x}; \quad \mathbf{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}; \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \)

**Solution:** In order to verify that the given vectors are solutions, we substitute \( \mathbf{x} \) as \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) in the equation \( \mathbf{x}' = A \mathbf{x} \) where \( A \) is the given matrix and show that the left hand side and right hand side match:

(i) \( \mathbf{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \Rightarrow \mathbf{x}_1' = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} e^{3t} \)

and

\[ A \mathbf{x}_1 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} e^{3t}. \]

Clearly, the two are equal and thus, it is verified that \( \mathbf{x}_1 \) is a solution of the given system.

(ii) Repeat the above procedure for \( \mathbf{x}_2 \) to get both \( \mathbf{x}_2' \) and \( A \mathbf{x}_2 \) to both be equal to

\[ \begin{bmatrix} -4e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} e^{-2t}. \]

The Wronskian is the determinant of the matrix formed by columns \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \):

\[ W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^t, \]

which is never equal to zero since \( e^t \) is always positive. Thus, the given solutions are linearly independent.

The general solution is given by

\[ \mathbf{x}(t) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \]

where \( c_1 \) and \( c_2 \) are constants. Plugging in \( t = 0 \) in the above equation and using the initial condition we have

\[ \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 3c_1 + c_2 \end{bmatrix}. \]

Therefore, to find \( c_1 \) and \( c_2 \) we need to solve the system of linear equations:

\[ \begin{align*} c_1 + 2c_2 &= 0 \\ 3c_1 + c_2 &= 5 \end{align*} \]

This gives us \( c_1 = 2, c_2 = -1 \). Thus, a particular solution is

\[ \mathbf{x}(t) = 2e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} - e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]
(b) \( \mathbf{x}' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix} \mathbf{x}; \quad \mathbf{x}_1 = e^{-2t} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = e^{3t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \)
\[
\mathbf{x}(0) = \begin{bmatrix} 5 \\ -7 \\ 11 \end{bmatrix}
\]

**Solution:** Verify by yourself that the given vectors indeed solve the linear system. The procedure is the same as described above, with column vectors of size 3 instead of 2.

The Wronskian is given by
\[
W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{vmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{vmatrix} = e^{2t},
\]
which is non-zero for every \( t \in \mathbb{R} \). The general solution is given by
\[
\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]

To find the values of the constants, we use the given initial condition to write
\[
\begin{bmatrix} 5 \\ -7 \\ 11 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} 3c_1 + c_2 + c_3 \\ -2c_1 - c_2 - c_3 \\ 2c_1 + c_2 + 0 \end{bmatrix}.
\]

This results in requiring to solve the linear system of equations:
\[
\begin{align*}
3c_1 + c_2 + c_3 &= 5 \\
-2c_1 - c_2 - c_3 &= -7 \\
2c_1 + c_2 + 0 &= 11.
\end{align*}
\]

Using (c) in (b) we get \( c_3 = -4 \). Using this in (a) and (c) we get \( c_1 = -2, \ c_2 = 15 \). Thus, a particular solution of the given system is
\[
\mathbf{x}(t) = -2e^{-2t} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} + 15e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 4e^{3t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]

2. Use the eigenvalue method to find a general solution of the given system.

(a) \( \mathbf{x}'_1 = 4\mathbf{x}_1 + \mathbf{x}_2 + 4\mathbf{x}_3, \quad \mathbf{x}'_2 = \mathbf{x}_1 + 7\mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{x}'_3 = 4\mathbf{x}_1 + \mathbf{x}_2 + 4\mathbf{x}_3. \)

**Solution:** The given system can be written as \( \mathbf{x}' = A \mathbf{x} \):
\[
\mathbf{x}' = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \mathbf{x},
\]

where
\[
\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.
\]
We first find the eigenvalues:

\[ |A - \lambda I| = 0 \implies \begin{vmatrix} 4 - \lambda & 1 & 4 \\ 1 & 7 - \lambda & 1 \\ 4 & 1 & 4 - \lambda \end{vmatrix} = 0 \]

which on simplifying gives the equation

\[-\lambda^3 + 15\lambda^2 - 54\lambda = -\lambda(\lambda^2 - 15\lambda + 54) = 0.\]

i.e., \(-\lambda(\lambda - 6)(\lambda - 9) = 0\), so the eigenvalues are \(\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 9\). Compute eigenvectors \(v_1, v_2, v_3\) for each eigenvalue by solving the equation \((A - \lambda_i I) v_i = 0\). One computation is illustrated below:

Consider \(\lambda_2 = 6\). We need to find \(v_2\) such that \((A - \lambda_2 I) v_2 = 0\), i.e.,

\[
\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

After performing Gaussian elimination on the coefficient matrix, we get

\[
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\]

from which we may conclude that

\(v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\)

is an eigenvector for \(\lambda_2\). The vectors \(v_1, v_3\) can be found similarly. In conclusion, the eigenvectors are:

\(v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\).

(Note: constant multiples of eigenvectors also describe the same eigenvector, so you might get a constant times the eigenvector recorded above, and that's still correct!) It follows that the general solution of our DE system is given by

\[ x(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

(b) \(x'_1 = x_1 + 2x_2 + 2x_3, \quad x'_2 = 2x_1 + 7x_2 + x_3, \quad x'_3 = 2x_1 + x_2 + 7x_3.\)

Solution: Proceed in the same way as in part (a). The matrix is

\[
A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}.
\]

The eigenvalues are once more 0, 6, 9 and the corresponding eigenvectors this time are

\(v_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}. \)
Thus, the general solution is given by
\[ x = c_1 \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{9t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}. \]

Alternatively, the solution is given by
\[ x_1(t) = -4c_1 + c_3 e^{9t}, \quad x_2(t) = c_1 - c_2 e^{6t} + 2c_3 e^{9t}, \quad x_3(t) = c_1 + c_2 e^{6t} + 2c_3 e^{9t}. \]

(c) \( x_1' = 4x_1 + x_2, \quad x_2' = 6x_1 - x_2 \)

**Solution:** Proceed as in part (a). The eigenvalues are \(-2, 5\) with eigenvectors \(v_1 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}\) and \(v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) respectively. Thus, the general solution is given by
\[ x = c_1 e^{-2t} \begin{pmatrix} 1 \\ -6 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

Alternatively,
\[ x_1(t) = c_1 e^{-2t} + c_2 e^{5t}, \quad x_2(t) = -6c_1 e^{-2t} + 2c_2 e^{5t}. \]

3. Solve the initial value problem \( x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1. \)

**Solution:** The given system can be written as
\[ x' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} x; \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \]

We first find the eigenvalues of the matrix \( A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \), i.e., we find \( \lambda \) such that \( |A - \lambda I| = 0 \):
\[ |A - \lambda I| = \begin{vmatrix} 3 - \lambda & 4 \\ 3 & 2 - \lambda \end{vmatrix} = (3 - \lambda)(2 - \lambda) - 12 = \lambda^2 - 5\lambda - 6 = 0. \]

The roots of \( \lambda^2 - 5\lambda - 6 = 0 \) are the eigenvalues: \(-1, 6\).

The eigenvalues are real and distinct. So we just have to find eigenvectors:
(i) Eigenvector for \( \lambda = -1 \):
\( (A - \lambda I) v_1 = 0 \implies (A + I) v_1 = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \)

This gives us the equation \( 4a + 4b = 0 \implies a = -b \). We may choose \( b \) to be any value and obtain the value of \( a \) using the condition \( a = -b \). For simplicity, pick \( b = -1 \). Thus,
\[ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

(ii) Eigenvector for \( \lambda = 6 \):
\( (A - \lambda I) v_2 = 0 \implies (A - 6I) v_2 = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \)

This gives us the equation \(-3a + 4b = 0 \implies a = \frac{4}{3} b \). Choosing \( b = 3 \) gives \( a = 4 \). Thus,
\[ v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}. \]
Therefore, the general solution is given by $x = c_1e^{-t}v_1 + c_2e^{6t}v_2$:

$$x = c_1e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{6t} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1e^{-t} + 4c_2e^{6t} \\ -c_1e^{-t} + 3c_2e^{6t} \end{bmatrix}.$$  

Alternatively, we express the solution by writing the components of the column vectors:

$$x_1(t) = c_1e^{-t} + 4c_2e^{6t}, \quad x_2(t) = -c_1e^{-t} + 3c_2e^{6t}$$

We find the constants $c_1, c_2$ using the initial condition $x_1(0) = 1, x_2(0) = 1$, giving us the equations:

$$c_1 + 4c_2 = 1$$
$$-c_1 + 3c_2 = 1.$$  

solving which gives us $c_1 = -\frac{1}{7}$ and $c_2 = \frac{2}{7}$. Therefore, the desired solution is

$$x_1(t) = \frac{1}{7}(-e^{-t} + 8e^{6t}), \quad x_2(t) = \frac{1}{7}(e^{-t} + 6e^{6t}).$$

4. Consider a cascade of two tanks where the volume of brine water in the first tank is 25 liters and the volume in the second tank is 40 liters. Each tank initially contains 15 kg of salt. Fresh water is flowing in the first tank, and well-mixed brine water is flowing out of the first tank into the second one and out of the second one. The three flow rates are all 5 liters/min. Find the amount of salt present in each tank after $t$ min.

**Solution:** Let $x_i(t)$ be the amount of salt in Tank $i$ after $t$ minutes for $i = 1, 2$. The linear system modelling the situation are

$$x'_1 = -\frac{5}{25}x_1$$
$$x'_2 = \frac{5}{25}x_1 - \frac{5}{40}x_2,$$

with coefficient matrix

$$\begin{bmatrix} -1/5 & 0 \\ 1/5 & -1/8 \end{bmatrix}.$$  

The characteristic equation is

$$(-1/5 - \lambda)(-1/8 - \lambda) = 0,$$

so the eigenvalues are $\lambda = -1/5$ and $\lambda = -1/8$. We then find that $\begin{bmatrix} 3 \\ -8 \end{bmatrix}$ is an eigenvector for $\lambda = -1/5$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1/8$, therefore the general solution to the linear system is

$$x_1(t) = 3c_1e^{-t/5}$$
$$x_2(t) = 8c_1e^{-t/5} + c_2e^{-t/8}.$$

The initial conditions now say

$$3c_1 = 15$$
$$-8c_1 + c_2 = 15,$$

from which we find $c_1 = 5, c_2 = 55$. Thus,

$$x_1(t) = 15e^{-t/5}$$
$$x_2(t) = -40e^{-t/5} + 55e^{-t/8}.$$  

As usual, we omitted many details of the calculations here. Make sure you can carry them out!
5. Consider the multiple brine tank setup illustrated below (picture omitted). The volume of Tank 1 is \( V_1 = 100 \) gallons, the volume of Tank 2 is \( V_2 = 50 \) gallons, and the rate of flow between the tanks is \( r = 20 \) gallons per minute. Suppose that initially, Tank 1 contains 15 pounds of salt and Tank 2 contains 0 pounds of salt. Find the amount of salt in each tank after \( t \) minutes.

**Solution:** Let \( x_i(t) \) be the amount of salt in Tank \( i \) after \( t \) minutes. The linear system modelling the situation are

\[
\begin{align*}
x'_1 &= -\frac{20}{100}x_1 + \frac{20}{50}x_2 \\
x'_2 &= \frac{20}{100}x_1 - \frac{20}{50}x_2,
\end{align*}
\]

with coefficient matrix

\[
\begin{bmatrix}
-0.2 & 0.4 \\
0.2 & -0.4
\end{bmatrix}.
\]

The characteristic equation is

\[(-0.2 - \lambda)(-0.4 - \lambda) - 0.08 = \lambda^2 + 0.6\lambda,\]

so the eigenvalues are 0 and \(-0.6\). We can then find that \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is an eigenvector for \( \lambda = 0 \) and \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) is an eigenvector for \( \lambda = -0.6 \), therefore the general solution to the linear system is

\[
\begin{align*}
x_1(t) &= 2c_1 - c_2 \cdot e^{-0.6t}, \\
x_2(t) &= c_1 + c_2 e^{-0.6t}.
\end{align*}
\]

The initial conditions now say

\[2c_1 - c_2 = 15, \quad c_1 + c_2 = 0,\]

so \( c_1 = 5, c_2 = -5 \). Thus,

\[
\begin{align*}
x_1(t) &= 10 + 5e^{-0.6t}, \\
x_2(t) &= 5 - 5e^{-0.6t}.
\end{align*}
\]