Additive group invariants in positive characteristic

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(joint with A. Maurischat)

- $\mathbf{k} = \overline{\mathbf{k}}$ is an algebraically closed field.
- B is a finitely generated $\mathbf{k}$-algebra.
- G is an algebraic group acting on B via $\mathbf{k}$-algebra automorphisms.
- Finally, B^G, the ring of invariants, is the subring of B fixed by the action of G.

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A central question in invariant theory:

Is B^G a finitely generated $\mathbf{k}$-algebra?

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  - If G is not reductive, there always exists B with a G-action such that B^G is not finitely generated. (Popov, 1979)

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Now, we concentrate on $G = \mathbb{G}_a$ and $B = \mathbf{k}[x_1, \ldots, x_n]$.

Additive group actions in characteristic zero

In characteristic zero:

- Maurer-Weitzenböck Theorem (1932):
  Linear $\mathbb{G}_a$-actions have finitely generated invariants.

- If $n \leq 3$, then the invariants are finitely generated. (Zariski, 1954)

- If $n \geq 5$, the invariants may not be finitely generated:
  - $n = 7$: Roberts's counterexample (1990)
  - $n = 6$: Freudenburg's counterexample (2000)
  - $n = 5$: Daigle and Freudenburg's counterexample (1999)

- The case $n = 4$ remains open for general $\mathbb{G}_a$-actions.

Additive group actions in positive characteristic

In positive characteristic:

- If $n \leq 3$, then the invariants are finitely generated. (Zariski, 1954)
- The invariants of some linear actions are finitely generated:
  - Basic actions (Seshadri, 1962)
  - Codimension-1 modules (Fauntleroy, 1977)
- It is not known if linear actions have finitely generated invariants.
- There are no known examples of $\mathbb{G}_a$-actions on polynomial rings with infinitely generated invariants.

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In this talk:
we construct positive characteristic analogs to Daigle and Freudenburg's, Freudenburg's, and Roberts's counterexamples, and show that in in every pisitive characteristic the invarinat rings are finitely generated.

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An additional structure

- In characteristic zero, additive group actions correspond to locally nilpotent derivations (LND).
  - It provides useful extra structure:
    for example, van den Essen's algorithm.
- This correspondance fails in positive characteristic.
- But there is a notion which plays the role of LND:
  Locally finite iterative higher derivations (lfihd).

Definition 1. A family $\left(\theta^{(i)}\right)_{i\geq 0}$ of $\mathbf{k}$-linear maps $B \longrightarrow B$ is a locally finite iterative higher derivation (lfihd) if and only if it fulfills the following properties:

1. $\theta^{(0)} = \mathrm{id}_B$,
2. for all $l \geq 0$ and $a, b \in B$, one has $\theta^{(l)}(ab) = \sum_{i+j=l} \theta^{(i)}(a)\theta^{(j)}(b)$,
3. for all $j, k \geq 0$ and $b \in B$, one has $\theta^{(j)}(\theta^{(k)}(b)) = \binom{j+k}{j}\theta^{(j+k)}(b)$,
4. for all $b \in B$, there is $l \geq 0$ such that $\theta^{(j)}(b) = 0$ for all $j \geq l$.

The ring of constants is

$$B^\theta := \{b \in B \mid \theta^{(i)}(b) = 0, \text{ for all } i \geq 1\}.$$

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Note that in characteristic zero: $\theta^{(j)} = \frac{1}{j!}(\theta^{(1)})^j$.

A $\mathbb{G}_a$-action on B corresponds to a $\mathbf{k}$-algebra automorphism

$$\theta : B \longrightarrow B \otimes_{\mathbf{k}} \mathbf{k}[U] = B[U]$$

via $\theta(b)|_{U=\sigma} = \sigma \cdot b,$ for $b \in B$.

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- $\left(\theta^{(i)}\right)_{i\geq 0}$ is a lfihd.
- The ring of invariants is equal to the ring of constants: $B^{\mathbb{G}_a} = B^{\theta}$.

Daigle and Freudenburg's counterexample (DF5)

$B_5 := \mathbf{k}[x, y, s, t, u, v]$ and in characteristic 0:

$$\theta^{(1)} = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v}.$$

To get something which works in all characteristics, we rescale the variables: $t := 2t, u := 6u$.

The lfihd is then given by

$$\begin{aligned}
\theta(x) &= x, \\
\theta(s) &= s + x^3 U, \\
\theta(t) &= t + 2sU + x^3 U^2, \\
\theta(u) &= u + 3tU + 3sU^2 + x^3 U^3, \\
\theta(v) &= v + x^2 U.
\end{aligned}$$

Theorem 2.
Contrary to characteristic zero, in every positive characteristic, the ring of invariants B_5^θ is finitely generated.

Proof.

As the lfihd θ is triangular, it restricts to a lfihd θ on $A_5 = \mathbf{k}[x, s, t, u]$.

A_5^θ is finitely generated. (easy, van den Essen's algorithm)

$\square$

Proof of Theorem 2 (continued).

There is an invariant of the form $v^p + vb' - b$. (the hard step!)

- van den Essen already had a sequence $xv^n + \ldots$ for all n
- when $n = p$, p divides the coefficients that are not divisible by x.

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- van den Essen already had a sequence $xv^n + \ldots$ for all n
- when $n = p$, p divides the coefficients that are not divisible by x.

The extension $A_5[v^p + vb' - b] \subset B_5$ is integral, hence so is the extension $A_5[v^p + vb' - b]^\theta \subset B_5^\theta$.

Thus, as $A_5[v^p + vb' - b]^\theta = A_5^\theta[v^p + vb' - b]$ is finitely generated, B_5^θ is finitely generated. $\square$

Robert's example: $B_7 = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3, v]$, with a lfihd θ
Freudenburg's example: $B_6 = \mathbf{k}[x, y, s, t, u, v]$, with a lfihd θ.

Theorem 3.
Contrary to characteristic zero, in every positive characteristic, the rings of invariants B_5^θ, B_6^θ, and B_7^θ are finitely generated.

Remark 4. Kurano (1993) did Roberts's example (B_7).

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Lemma 5 (The examples are closely related)**.**

1. $B_5 \cong B_6/(y-1)$ *(respects the lfihd)*
2. $\alpha : B_6 \to B_7$ *a homomorphism respecting the lfihd*

[1] Emilie Dufresne and Andreas Maurischat. On the finite generation of additive group invariants in positive characteristic. *accepted for publication in J. Algebra*, 2010.