On the depth of separating algebras of finite groups

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Outline

1. Basic Definitions
   - Invariant Theory
2. Separating Algebras
   - Our results
   - Example

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1 Basic Definitions
- Invariant Theory

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Cast List

- G, a group
- k a field of characteristic $p \geq 0$
- V, a finite-dimensional k-vector space on which G acts
- G acts on $k[V]$ via $\sigma.f(v) = f(\sigma^{-1}v)$

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The Ring of Invariants, $k[V]^G$ is the ring of fixed points under this action.

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Separating Invariants

A polynomial $f \in k[V]$ is said to separate the points v and w if $f(v) \neq f(w)$.

Theorem

Let G be a finite group and $v, w \in V$. Then the following are equivalent:

- *v and w lie in the same G-orbit*
- *There exists some $f \in k[V]^G$ separating v and w.*

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Separating Sets

Definition

A separating set is a set S of invariants with the following property: If $v, w \in V$ are separated by invariants, there exists $f \in S$ such that $f(v) \neq f(w)$.

Many results about generating sets of rings of invariants in characterstic zero extend to separating sets in arbitrary characteristic.

Suppose G is finite.

- In characteristic zero, the ring of invariants $k[V]^G$ is generated as a k-algebra by a set of invariants of degree $\leq |G|$.
- In prime characteristic, this is no longer true.
- In arbitrary characteristic, there exists a separating set consisting of invariants of degree $\leq |G|$.

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Depth and the Cohen-Macaulay property

- An element $f \in k[V]^G$ is called regular on $k[V^G]$ if the map $k[V]^G \to k[V]^G$ defined by multiplication with f is injective.
- A sequence $f_1, \cdots f_r$ is called a regular sequence if f_i is regular on $k[V^G]/(f_1, \cdots, f_{i-1})$.
- If $I \subset k[V]^G$ is an ideal, then the length of the longest regular sequence in I is called the depth of I. The depth of $k[V^G]$ is defined to be the depth of the maximal ideal $k[V]^G_+$ generated by positive degree invariants.
- $k[V]^G$ is called Cohen-Macaulay if its depth and dimension are equal. In this case, the depth of any ideal is equal to its height. In fact, we have

$$\mathrm{depth}(k[V]^G) \leq \dim(k[V]^G) + \mathrm{depth}(I) - \mathrm{height}(I)$$

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Separating algebras are very closely related to rings of invariants:

Theorem (Derksen, Kemper)

Suppose k is an algebraically closed field of characteristic $p > 0$. Let $A \subset k[V]^G$ be a separating algebra. Then

- $$k[V^G] = \{f \in k[V] : f^{p^m} \in A \text{ for some } m \geq 0\}$$
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- $k[V]^G$ *is an integral extension of A.*

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Let $A \subset k[V]^G$ be a separating algebra:

Question: Is $\mathrm{depth}(A) \leq \mathrm{depth}(k[V]^G)$?

To answer this, we recall a method of calculating $\mathrm{depth}(k[V]^G)$.

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Theorem (Kemper,1999)

Let $0 \neq \alpha \in H^m(G, k[V])$ *where* $m := \min\{i : H^i(G, k[V]) \neq 0\}$. *Let* $I := \mathrm{Ann}_{k[V]^G}(\alpha)$. *Then*

$$\mathrm{depth}(I) = \min\{m+1, \mathrm{height}(I)\}$$

Theorem

Let $\chi_\alpha := \{X \leq G : \mathrm{res}^G_X(\alpha) \neq 0\}$. *Then*

$$\mathrm{height}(I) = \min\{\dim(V^X) : X \in \chi_\alpha\}.$$

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Theorem (Dufresne, E, Kohls ,2009)

Let $0 \neq \alpha \in H^m(G, k[V])$ be such that α^{p^r} is nonzero for all r and m is the smallest index for which such an α exists. Let $I := \mathrm{Ann}_A(\alpha)$. Then

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Our main result is the following:

Theorem

Suppose G and V are such that one of the following holds:

- *G is p-nilpotent with a cyclic Sylow-p-subgroup P,*
- *(G, V) is a shallow group,*
- *V is a permutation module, and $|G|$ is not divisible by p^2.*

Suppose $A \subset k[V]^G$ is a separating algebra. Then $\mathrm{depth}(A) \leq \mathrm{depth}(k[V]^G)$.

Let $G := \mathbb{Z}/2 \times \mathbb{Z}/2$ and let k be an algebraically closed field of characteristic 2. Let V be an indecomposable kG-module. Let Q_1, Q_2, Q_3 be the nontrivial proper subgroups of G with $\dim(V^{Q_1} \leq \dim(V^{Q_2}) \leq \dim(V^{Q_3}))$. Let $A \subset k[V]^G$ be a separating algebra.

Theorem (E, 2007)

Provided V is not projective

$$\mathrm{depth}(k[V]^G) = \max\{\dim(V^G) + 2, \dim(V)\}.$$

Theorem (E, 2010)

$$\mathrm{depth}(A) \leq \dim(V^{Q_2}) + 2.$$

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The indecomposable representations of G are classified into certain families. It turns out that for all V of even dimension we have $\mathrm{depth}(A) \leq \mathrm{depth}(k[V]^G)$. In each odd dimension, there are two non-isomorphic kG-modules, one of which satisfies $\mathrm{depth}(A) \leq \mathrm{depth}(k[V]^G)$, while for the other only the weaker statement $\mathrm{depth}(A) \leq \mathrm{depth}(k[V]^G) + 1$ holds.

Of these modules, only one could possibly provide an example of a non-Cohen-Macaulay ring of invariants containing a Cohen-Macaulay separating algebra.

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Consider the 5-dimensional representation of G given by

$$\sigma \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \tau \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The depth of the ring of invariants is 4. A separating set is given by the following seven polynomials:

$$\begin{aligned} \{ & a_1 := x_3, a_2 := x_4, a_3 := x_5, \\ & a_4 := x_1^4 + x_1^2 x_3^2 + x_1^2 x_3 x_4 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_1 x_3 x_4 x_5 + \\ & \qquad x_1 x_4^3 + x_2^2 x_3^2 + x_2 x_3^2 x_5 + x_2 x_3 x_4^2, \\ & a_5 := x_2^4 + x_2^2 x_4^2 + x_2^2 x_4 x_5 + x_2^2 x_5^2 + x_2 x_4^2 x_5 + x_2 x_4 x_5^2, \\ & a_6 := x_1^2 x_4^2 + x_1 x_3 x_4 x_5 + x_1 x_4^3 + x_2^2 x_3^2 + x_2 x_3^2 x_5 + x_2 x_3 x_4^2, \\ & a_7 := x_1 x_4^2 x_5 + x_1 x_4 x_5^2 + x_2^2 x_3 x_5 + x_2^2 x_4^2 + x_2 x_3 x_5^2 + x_2 x_4^3 \} \end{aligned}$$

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