Locally Nilpotent Derivations of Rings with Roots Adjoined

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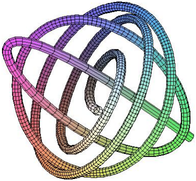

Joint work with Lucy Moser-Jauslin, Universite de Bourgogne

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k is a field of characteristic 0, and R is a commutative **k**-algebra. For a ring B, $\mathrm{LND}(B)$ is the set of locally nilpotent derivations D of B. These correspond to the $\mathbb{G}_a$-actions on $\mathrm{Spec}(B)$.

Main Idea. Study $\mathrm{LND}(B)$ when $B = R[z]$ is a domain, and $z^n \in R$ for $n \geq 2$.

In this case, B is a free R-module:

$$B = R + Rz + \cdots + Rz^{n-1}$$

This becomes a $\mathbb{Z}_n$-**grading** of B over R when $\deg z \in \mathbb{Z}_n^*$. It is natural to consider LNDs of B which are *homogeneous* relative to this grading.

Basic Theorem.

Let $f \in R$ be such that $B = R[z]/(f + z^n)$

Theorem 1. Let $D \in \mathrm{LND}(B)$ be $\mathbb{Z}_n$-homogeneous.

(a) $D|_R = z^\lambda \delta$ for some $\delta \in \mathrm{LND}(R)$ and $0 \leq \lambda \leq n-1$

(b) $Dz, \delta f \in \ker \delta = R \cap \ker D$

(c) If $Dz \neq 0$, then $\lambda = n-1$ and $\ker D = \ker \delta$

Definition. Given non-zero $D \in \mathrm{LND}(R)$, if $K = \mathrm{frac}(\ker D)$ and $t \in R$ is a local slice, then $R \subset K[t] = K^{[1]}$. Therefore, D defines a degree function on R. The **absolute degree** of $f \in R$ is the minimal degree of f over all such D, denoted $|f|_R$. In case $\mathrm{LND}(R) = \{0\}$ (R is **rigid**), set $|f|_R = \infty$ for $f \neq 0$.

Note. Part (b) of the theorem says $|f|_R \leq 1$. We therefore obtain:

Corollary 1. R is $\mathbb{Z}$-graded and affine, $f \in R$ is homogeneous ($f \neq 0$), $n \geq 2$ and $\gcd(n, \deg f) = 1$. TFAE:

(a) $|f|_R \geq 2$

(b) $B = R[z]/(f + z^n)$ is rigid

Corollary 2. R is a rigid affine $\mathbf{k}$-domain and $f \in R$ is non-zero. Given relatively prime $m, n \geq 1$, the ring

$$B = R[x, y]/(f + x^m y^n) \ .$$

is rigid.

In order to apply these results, we first need to calculate some absolute degrees, which is generally difficult if R is not rigid.

Proposition 1. If $R = \mathbf{k}[x, y] = \mathbf{k}^{[2]}$ and $a, b \geq 2$, then

$$|x^a + y^b|_R \geq \begin{cases} \min\{a, b\} & a \neq b \\ a - 1 & a = b \end{cases},$$

with equality when $\mathbf{k}$ is algebraically closed.

Proposition 2. (due to Daigle) If $R = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$ and $a, b, c \geq 2$, where at most one of a, b, c equals 2, then

$$|x^a + y^b + z^c|_R \geq 2 .$$

Application: Pham-Brieskorn Surfaces

The Pham-Brieskorn surfaces are defined by $S = \mathrm{Spec}(B)$ for

$$B = \mathbf{k}[x, y, z]/(x^a + y^b + z^c) \ ,$$

where $a, b, c \geq 2$.

Theorem 2. If at most one of a, b, c equals 2, then B is rigid.

Proof. If $\mathrm{LND}(B) \neq \{0\}$, then $B \subset K[t]$ as above, and

$$x(t)^a + y(t)^b + z(t)^c = 0 \quad \text{in} \quad K[t] \ .$$

By Mason's Theorem, $1/a + 1/b + 1/c > 1$, meaning (a, b, c) equals

$$(2,3,3) \ , \ (2,3,4) \ , \ \text{or} \ (2,3,5)$$

in some order. We may thus assume $\gcd(ab, c) = 1$. Set

$$R = \mathbf{k}[x, y] \quad \text{and} \quad f = x^a + y^b \ .$$

Then R is $\mathbb{Z}$-graded with $\deg x = b$, $\deg y = a$, and $\deg f = ab$. Since $|f|_R \geq 2$, the ring $B = R[z]/(f + z^c)$ is rigid by Cor.2, a contradiction. Therefore, $\mathrm{LND}(B) = \{0\}$. $\square$

Pham-Brieskorn Threefolds.

Suppose $a, b, c \geq 2$ and at most one of these integers equals 2. If $R = \mathbf{k}[x, y, z]$, then by Daigle's result, $|x^a + y^b + z^c|_R \geq 2$. It follows from Cor. 1 that, for any $d \geq 2$ with $\gcd(abc, d) = 1$, the ring

$$B = \mathbf{k}[x, y, z, t]/(x^a + y^b + z^c + t^d)$$

is rigid.

Open Question. Is the affine Fermat cubic threefold

$$x^3 + y^3 + z^3 + t^3 = 0$$

rigid?

Stable Rigidity.

Definition. If A is an affine **k**-domain, then A is **stably rigid** if, for all $n \geq 0$,

$$\mathrm{LND}(A^{[n]}) = \mathrm{LND}_A(A^{[n]}) \ .$$

If A is stably rigid, then it is rigid, but it is an open question whether the converse holds.

Makar-Limanov has shown:

- If A is rigid, then $\mathrm{LND}(A^{[1]}) = \mathrm{LND}_A(A^{[1]})$.
- If A is rigid and $\dim A = 1$, then A is stably rigid.

What about the Pham-Brieskorn surfaces? When

$$1/a + 1/b + 1/c \leq 1 \ ,$$

the proof above shows $B = \mathbf{k}[x, y, z]/(x^a + y^b + z^c)$ is **stably rigid**.

Question. Let

$$B = \mathbf{k}[x, y, z]/(x^2 + y^3 + z^5)$$

which is a UFD, and let $D \in \mathrm{LND}(B[X, Y])$. Does $DB = 0$?

Pham-Brieskorn Surfaces with Parameters.

Note that for *any* $a, b, c \geq 2$, the ring

$$B = \mathbf{k}[t, x, y, z]/(tx^a + y^b + z^c)$$

is non-rigid. However, we have:

Theorem 3. Given $a, b, c, d \geq 2$, let $e = \gcd(a, d)$, and

$$B = \mathbf{k}[t, x, y, z]/(t^d x^a + y^b + z^c) \ .$$

If at most one of e, b, c equals 2, then B is rigid.

Proof. By Thm. 2, the ring $R = \mathbf{k}[y, z, v]/(v^e + y^b + z^c)$ is rigid. Therefore, if $a = em$ and $d = en$, then Cor. 2 implies that

$$B = R[t, x]/(v - t^n x^m)$$

is rigid. $\square$

Note. We have similar results for

$$B = \mathbf{k}[t, x, y, z]/(t^d x^a + t^e y^b + z^c) \ .$$

These threefolds have been studied by Kaliman and Makar-Limanov.

A Second Basic Theorem

- $f \in R$ and $m, n \geq 2$ are relatively prime
- $B = R[y, z]/(f + y^m + z^n)$
- $S = R[y] \cap R[z]$

Then B is a free S-module:

$$B = \oplus S y^i z^j \quad (0 \leq i \leq m-1\,,\, 0 \leq j \leq n-1)\ .$$

This yields a $\mathbb{Z}_{mn}$-grading of B over S when

$$\deg y = un\ (u \in \mathbb{Z}_m^*) \quad \text{and} \quad \deg z = vm\ (v \in \mathbb{Z}_n^*)\ .$$

Theorem 4. Let $D \in \mathrm{LND}(B)$ be $\mathbb{Z}_{mn}$-homogeneous.

(a) $D^2 y = D^2 z = 0$

(b) $Dy = 0$ or $Dz = 0$

Application. Let A be a commutative **k**-domain and $B = A[x_1, \dots, x_n] = A^{[n]}$. Suppose B is $\mathbb{Z}$-graded over A, where $\deg x_i = a_i$ and $\gcd(a_1, \dots, a_n) = 1$.

Theorem 5. Suppose $D \in \mathrm{LND}(B)$ is $\mathbb{Z}$-homogeneous.

(a) If $\gcd(a_1, \dots, \hat{a}_i, \dots, a_n) \neq 1$, then $D^2 x_i = 0$.

(b) If $\gcd(a_1, \dots, \hat{a}_i, \dots, a_n) \neq 1$ and $\gcd(a_1, \dots, \hat{a}_j, \dots, a_n) \neq 1$ for $i \neq j$, then $Dx_i = 0$ or $Dx_j = 0$.

In other words, we get information about *all* homogeneous LNDs just from numerical data! Note that we do not assume $DA = 0$.

Concluding Remarks.

1. Fields are rigid, and in this sense, rigid rings are generalizations of fields.

2. We have focused on rings $R[z]$ such that $z^n \in R$. There are 3 related classes of rings which naturally suggest themselves for similar investigation.

- Rings of the form $R[z]$ where z is integral over R
- Rings of the form $R[z]$ where $z \in \mathrm{frac}(R)$
- Rings of the form $R[x,y]$ where $xy \in R$.

Two basic results:

- If $a, b \in R$ are non-zero, and $|a|_R = 0$, then $R[z]/(az+b)$ is not rigid.
- (from Cor. 2) If R is rigid and $f \in R$ is non-zero, then $R[x,y]/(f+xy)$ is rigid.