Describing Resolvent Sextics of Quintic Equations

By Frank D. Grosshans

L. E. Dickson, Bulletin A.M.S. 31, 1925, 515 - 523

RESOLVENT SEXTICS OF QUINTIC EQUATIONS

BY L. E. DICKSON

1. *Introduction*. The object of this paper is to give simple derivations of the classic resolvents which have been obtained heretofore by elaborate computations.

Jacobi* established the form of a remarkable resolvent, but neither found the values of the coefficients nor gave the simple details (§2 below) which lead directly to that form.

Cayley† was not aware of Jacobi's work when he fully computed the same resolvent. Noting that its roots are functions of the differences of the roots x_i of the quintic, he first computed at length the resolvent sextic under the restriction that $x_5 = 0$. Then the coefficients were "completed by the introduction of the terms involving the constant coefficient of the quintic." No details were given of the latter long computation, which may perhaps be best made by utilizing the fact that the coefficients are seminvariants. The simple new method employed here (§3) makes initial use of the latter fact as well as of a lemma which reduces the search for the needed seminvariants of the quintic to a mere inspection of the invariants of a quartic.

From the Jacobi-Cayley resolvent (which is a simple transform of the old Malfatti resolvent) it is an immediate step (§5) to the noteworthy covariant resolvent discovered by Perrin‡. and independently by McClintock,§ each time as the final step of a long computation.

* JOURNAL FÜR MATHEMATIK, vol. 13 (1835), pp. 340-52; WERKE, vol. 3, 1884, pp. 269-84.

† PHILOSOPHICAL TRANSACTIONS, London, vol. 151 (1861), pp. 263-76; COLLECTED MATHEMATICAL PAPERS, vol. 4, pp. 309-24.

‡ COMPTES RENDUS DU DEUXIÈME CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Paris, 1902, pp. 199-223. Announced in BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 11 (1882-83), pp. 64-65.

§ AMERICAN JOURNAL, vol. 8 (1886), pp. 45-84; vol. 20 (1898), pp. 157-192.

- §1. Resolvent program
- §2. Solving the cubic equation
- §3. Describing resolvent sextics for quintics
- §4. Ideas in proof
- §5. McClintock - Perrin resolvent

§1. Resolvent program

- Goal: to find formulas for roots of $f(z) = (z - x_1) \dots (z - x_n)$.

(R1) Find a polynomial $\phi \in F[x_1, \dots, x_n]$, a *resolvent*.

(R2) Find a polynomial in $F[x_1, \dots, x_n][z]$ having symmetric functions as coefficients and which has ϕ as a root, the *resolvent equation*.

(R3) Use information about ϕ to determine whether the original equation can be solved and, if so, solve it.

- (R2). Given $\phi \in F[x_1, \dots, x_n]$, find a polynomial in $F[x_1, \dots, x_n][z]$ having symmetric functions as coefficients and which has ϕ as a root.
- The Construction Principle. Let $\phi = \phi_1, \dots, \phi_m$ be all the images of ϕ under the symmetric group S_n. Put

$$R(z) = (z - \phi_1) \dots (z - \phi_m)$$

§2. Solving the cubic equation

- $f(z) = z^3 + bz^2 + cz + d = (z - x_1)(z - x_2)(z - x_3)$

- Vieta's formulas

$$-b = x_1 + x_2 + x_3$$

$$c = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$-d = x_1 x_2 x_3$$

- $\omega^3 = 1$, $\omega \neq 1$

- $\phi = x_1 + \omega\, x_2 + \omega^2\, x_3$
  $\psi = x_1 + \omega^2\, x_2 + \omega\, x_3$

- If know ϕ and ψ, can combine with
  $-b = x_1 + x_2 + x_3$
  (Vieta's formula) and obtain roots x_1, x_2, x_3 by solving system of linear equations.

- (R1) Find a polynomial $\phi \in F[x_1, \dots, x_n]$, a *resolvent*.

- ϕ has 6 images under the action of S_3:

$$\phi,\ \psi,\ \omega\,\phi,\ \omega\,\psi,\ \omega^2\,\phi,\ \omega^2\,\psi$$

- No help from Construction Principle (get 6th degree equation for ϕ).

- ϕ^3 has only 2 images, ϕ^3 and ψ^3.

- (R2) Find a polynomial in $F[x_1, \ldots, x_n][z]$ having symmetric functions as coefficients and which has ϕ^3 as a root, the *resolvent equation*.

- By Construction Principle,

$$(z - \phi^3)(z - \psi^3)$$

has coefficients which are symmetric functions, so polynomials in b, c, d.

- $\phi^3 + \psi^3 = -2b^3 + 9bc - 27d$
  $\phi^3 \psi^3 = (b^2 - 3c)^3$

- $(z - \phi^3)(z - \psi^3) = z^2 - (-2b^3 + 9bc - 27d)z + (b^2 - 3c)^3$

- (R3) Use information about ϕ to determine whether the original equation can be solved and, if so, solve it.

- Can solve quadratic equation above, then take cube roots carefully.

§3. Describing resolvent sextics for quintics

- F_{20} = subgroup of S_5 generated by the cycles (1 2 3 4 5) and (2 3 5 4).

- **Theorem.** Let $f(x) \in \mathbf{Q}[x]$ be an irreducible quintic. Then $f(x) = 0$ is solvable in radicals if and only if its Galois group is a subgroup of F_{20}.

- (R1) Find a polynomial $\phi \in F[x_1, \ldots , x_5]$, a *resolvent*.

- Translation from Galois theory: find $\phi(x_1, x_2, x_3, x_4, x_5)$ fixed by F_{20} but not by S_5.

- Φ_4 = Malfatti (1771), Jacobi (1835), Cayley (1861)

- $\Phi_4 = (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$

  $-x_1x_3 - x_3x_5 - x_5x_2 - x_2x_4 - x_4x_1)^2$

- ϕ as above, $p = p(x_1, x_2, x_3, x_4, x_5)$ a symmetric function, then $p + \phi$ and $p \times \phi$ have desired property. So, infinitely many resolvents.

- (Dummitt, 1991) : $x_1^2x_2x_5 + x_1^2x_3x_4 + \ldots$

- 4Dummitt - Malfatti = symmetric function

- **Theorem (new).** There are F_{20} invariants: Φ_4 of degree 4, Φ_5 of degree 5, Φ_6 of degree 6, Φ_7 of degree 7, Φ_8 of degree 8, so that any F_{20} invariant f can be written uniquely in the form

$$f = r_1 + r_2\Phi_4 + r_3\Phi_5 + r_4\Phi_6 + r_5\Phi_7 + r_6\Phi_8$$

where each r_i is a symmetric function.

- Φ_4 = Malfatti (1771), Jacobi (1835), Cayley (1861)

- Φ_5 = Lagrange (1771)

- (R2) Find a polynomial in $F[x_1, \ldots, x_n][z]$ having symmetric functions as coefficients and which has ϕ as a root, the *resolvent equation*.

- ϕ has six images under S_5, since $[S_5 : F_{20}] = 6$, say $\phi = \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$. The *resolvent equation* is

- $R(z, f) = (z - \phi_1)(z - \phi_2)(z - \phi_3)(z - \phi_4)(z - \phi_5)(z - \phi_6)$.

- **Theorem (new?).** Let ϕ be fixed by F_{20} but not by S_5 and form $R(z, f)$ as above. Let $f_o(x) \in \mathbf{Q}[x]$ be an irreducible quintic.

- (a) If $f_o(x) = 0$ is solvable in radicals, then $R(z, f_o)$ has a rational root.

- (b) If $R(z, f_o)$ has a rational root and does not have the form $(x - a)^6$, $a \in \mathbf{Q}$, then $f_o(x) = 0$ is solvable in radicals.

- Condition in (b) not restrictive for Jacobi and Lagrange resolvents.

- Rational root: for Jacobi resolvent, have algorithm to solve in radicals.

§4. Ideas in proof

- Find the invariants of $(\alpha\beta\gamma\delta\varepsilon)$ on ₵$[\alpha - \beta, \beta - \gamma, \gamma - \delta, \delta - \varepsilon]$.

- X, Y, Z, W eigenvectors of $(\alpha\beta\gamma\delta\varepsilon)$ on $< \alpha - \beta, \beta - \gamma, \gamma - \delta, \delta - \varepsilon >$ corresponding to z, z^2, z^3, z^4 where $z^5 = 1$.

- Find invariants of D_{10} , F_{20} on ₵$[\alpha, \beta, \gamma, \delta, \varepsilon]$.

- Malfatti: $\Phi_4 = (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 - x_1x_3 - x_3x_5 - x_5x_2 - x_2x_4 - x_4x_1)^2$
- $= (X\,W - Y\,Z)^2$
- $\Phi_5 = (X\,W - Y\,Z)(XY^2 + WZ^2 - ZX^2 - YW^2)$
- (505 monomials in x_1, x_2, x_3, x_4, x_5)

- $G = S_5$, $H = F_{20}$. $\mathbb{C}[\alpha, \beta, \gamma, \delta, \varepsilon]^H$ is free $\mathbb{C}[\alpha, \beta, \gamma, \delta, \varepsilon]^G$ - module, rank 6.

- J_G ideal generated by elementary symmetric functions. Find basis for $(\mathbb{C}[\alpha, \beta, \gamma, \delta, \varepsilon]/J_G)^H$. Gives basis for free module. (Chevalley's theorem on coinvariants essential.)

§5. McClintock - Perrin resolvent

- Change 1: allow rational functions as roots

$$R(f;x,y) = A\prod_{i=1}^{6}(a_0\Phi_i x - a_0\Psi_i y)$$

where

(1) Ψ_i, Φ_i are polynomials in α, β, γ, δ, ε

(2) $\frac{\Psi_1}{\Phi_1}$ fixed by F_{20}

(3) $\frac{\Psi_i}{\Phi_i} = \sigma_i \frac{\Psi_1}{\Phi_1}$, $S_5 = \cup\, \sigma_i F_{20}$

- Change 2: want "covariant", i.e.,

- $R(f; \begin{pmatrix} x \\ y \end{pmatrix}) = A_0 x^6 + A_1 x^5 y + \ldots + A_6 y^6$

  $R(g \cdot f\, ;\, g \cdot \begin{pmatrix} x \\ y \end{pmatrix}) = R(f; \begin{pmatrix} x \\ y \end{pmatrix})$ for all g in SL_2

- So, $\langle A_0, A_1, \ldots, A_6 \rangle$ representation of SL_2 isomorphic to that on binary sextics

- Properties of McClintock–Perrin resolvent

- (1) A_0 seminvariant, square is constant term in Jacobi resolvent

- McClintock-Perrin = $C_{51}C_{15} - 25C_{33}^2$

- (2) **Theorem.** Let $f_o(x) \in \mathbf{Q}[x]$, irreducible quintic . Then $f_o(x) = 0$ is solvable in radicals if and only if McClintock – Perrin resolvent has a rational root.

- (3) Having rational root, can solve quintic in "very elegant manner" (Cayley)

- **Theorem (new)**: McClintock resolvent is unique up to multiplication by SL_2 - invariant.

- **Theorem (new?)**: Solvable quintics are Zariski dense in space of all quintics.

Ideas in proof

- Use covariant property.
- $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$: Φ_1 homogeneous of degree 2
- $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$: $\Phi_1(x + t) = \Phi_1(x)$
- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: relates Φ_1 and ψ_1

- F_{20} relative invariant

- Relative F_{20} – invariant is absolute (12345) – invariant.

- McClintock parametrizes solvable quintics.

- Berndt, B. C., Spearman, B. K., Williams, K. S.: Commentary on an unpublished lecture by G. N. Watson on solving the quintic.
  www.math.uiuc.edu/~berndt/articles/watsonlecture.pdf
- Colin, Antoine Formal computation of Galois groups with relative resolvents. Applied algebra, algebraic algorithms and error-correcting codes (Paris, 1995), 169-182, Lecture Notes in Comput. Sci., 948, Springer, Berlin, 1995.
- Cox, David; Galois Theory. Wiley-Interscience, Hoboken, NJ 2004
- Dickson, L. E.; Resolvent sextics of quintic equations. Bull. Amer. Math. Soc. 31 (1925), no. 9-10, 515 - 523.
- Dickson, L. E.; Modern Algebraic Theories. Benj. H. Sanborn & Co. Chicago, Il. 1930

- Dummit, D. S. Solving solvable quintics. Math. Comp. 57 (1991), no. 195, 387 - 401.

  www.emba.uvm.edu/~dummit/quintics/solvable.pdf

- Girstmair, Kurt On invariant polynomials and their application in field theory. Math. Comp. 48 (1987), no. 178, 781 -797.

- McClintock, J. E.,; Analysis of quintics. Amer. J. Math. 8 (1886), 45 - 84.

- Tignol, Jean-Pierre Galois' theory of algebraic equations. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.