Invariants of a Vector and a Covector

Gregor Kemper (joint work with Cédric Bonnafé)

Fredericton, June 5, 2010

Doubly parametrized series of groups

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Dickson (1911)	folklore				
polynomial ring	polynomial ring				

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Dickson (1911)	folklore	Carlisle/ Kropholler (1992?)	Chu/Jow (2006)	?	Shank, Wehlau & others
polynomial ring	polynomial ring	complete intersection	complete intersection	complete intersecti- on?	wild

Doubly parametrized series of groups

What about decomposable representations?

Example. $V = \mathbb{F}_q^3$, $R = \mathbb{F}_q[V \oplus V]^{U_3(\mathbb{F}_q)}$:

$q =$	2	3	4,5, ...
generators:	12	16	?
structure:	Gorenstein	depth = 4	?

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We consider $\mathbb{F}_q[V \oplus V^*]^{U_n(\mathbb{F}_q)}$ $\quad \big(U_n(\mathbb{F}_q) = \{\text{upper unipotent matrices}\}\big)$.

$\rightarrow K[V \oplus V^*]^G$ useful for computing invariants in Weyl algebras.

The case $n = 2$

Set

$$\mathbb{F}_q[V \oplus V^*] = \mathbb{F}_q[x_1, x_2, y_1, y_2],$$

$$f_1 = x_1, \qquad g_1 = y_2,$$

$$f_2 = \prod_{h \in U_2(\mathbb{F}_q)\cdot x_2} h, \qquad g_2 = \prod_{h \in U_2(\mathbb{F}_q)\cdot y_1} h,$$

$$u_0 = x_1 y_1 + x_2 y_2 \quad \Big(\text{corresponding to } \mathrm{id}_V \in \mathrm{End}(V)\Big).$$

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Then

$$\mathbb{F}_q[V \oplus V^*]^{U_2(\mathbb{F}_q)} = \mathbb{F}_q[f_1, f_2, g_1, g_2, u_0]$$

subject to

$$u_0^q - (f_1 g_1)^{q-1} u_0 - f_1^q g_2 - g_1^q f_2 = 0.$$

The general case

Set

$$f_i = \prod_{h \in U_n(\mathbb{F}_q)\cdot x_i} h, \qquad g_i = \prod_{h \in U_n(\mathbb{F}_q)\cdot y_{n-i+1}} h \qquad (i = 1, \ldots, n),$$

$$u_j = \sum_{k=1}^{n} x_k^{q^j} y_k, \qquad u_{-j} = \sum_{k=1}^{n} x_k y_k^{q^j} \qquad (j \geq 0).$$

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Theorem (Bonnafé, Ke 2010):

$$\mathbb{F}_q[V \oplus V^*]^{U_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, \ldots, f_n, g_1, \ldots, g_n, u_{2-n}, \ldots, u_{n-2}]$$

($4n - 3$ generators), subject to $2n - 3$ relations.

This is a complete intersection!

The case $n = 3$

Relations:

$$u_{-1}^{q} - (g_1^{q(q-1)} + g_2^{q-1})u_0^q + (g_1 g_2)^{q-1} u_1 - f_1^q g_3 = 0, \qquad (R_1)$$

$$u_0^q - g_1^{q-1} u_1 - f_1^{q-1} u_{-1} + (f_1 g_1)^{q-1} u_0 - f_2 g_2 = 0, \qquad (R_2)$$

$$u_1^q - (f_1^{q(q-1)} + f_2^{q-1})u_0^q + (f_1 f_2)^{q-1} u_{-1} - f_3 g_1^q = 0. \qquad (R_3)$$

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- The invariant field is

$$\mathbb{F}_q(V \oplus V^*)^{U_3(\mathbb{F}_q)} = \mathbb{F}_q\left(f_1, f_2, f_3, g_1, g_2, g_3, u_0\right).$$

- It follows that $A \to \mathbb{F}_q[V \oplus V^*]^{U_n(\mathbb{F}_q)}$ is an isomorphism.

The invariant field

Let $G \subseteq \mathsf{GL}(V)$ be finite. Then $K(V \oplus V^*)^G$ is generated by:

- $K[V]^G$,
- $K[V^*]^G$, and
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The proof uses Galois theory.

Relations

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Lemma:

$$\sum_{i=1}^{k}\sum_{j=1}^{n+1-k}\sum_{l=1}^{n}(-1)^{i+j+n+1}a_{i,l}b_{j,n+1-l}\cdot\det\begin{pmatrix} a_{1,1} & \cdots & a_{1,k-1} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} \\ a_{i+1,1} & \cdots & a_{i+1,k-1} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} \end{pmatrix}$$

$$\cdot\det\begin{pmatrix} b_{1,1} & \cdots & b_{1,n-k} \\ \vdots & & \vdots \\ b_{j-1,1} & \cdots & b_{j-1,n-k} \\ b_{j+1,1} & \cdots & b_{j+1,n-k} \\ \vdots & & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n-k} \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{pmatrix}\cdot\det\begin{pmatrix} b_{1,1} & \cdots & b_{1,n+1-k} \\ \vdots & & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n+1-k} \end{pmatrix}.$$

Relations

Substituting $a_{i,j} = x_j^{q^{i-1}}$ and $b_{i,j} = y_{n+1-j}^{q^{i-1}}$ and setting

$$c_{k,i} := \det \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^q & x_2^q & \cdots & x_k^q \\ \vdots & \vdots & & \vdots \\ x_1^{q^{i-1}} & x_2^{q^{i-1}} & \cdots & x_k^{q^{i-1}} \\ x_1^{q^{i+1}} & x_2^{q^{i+1}} & \cdots & x_k^{q^{i+1}} \\ \vdots & \vdots & & \vdots \\ x_1^{q^k} & x_2^{q^k} & \cdots & x_k^{q^k} \end{pmatrix}, \quad d_{k,i} := \det \begin{pmatrix} y_n & y_{n-1} & \cdots & y_{n+1-k} \\ y_n^q & y_{n-1}^q & \cdots & y_{n+1-k}^q \\ \vdots & \vdots & & \vdots \\ y_n^{q^{i-1}} & y_{n-1}^{q^{i-1}} & \cdots & y_{n+1-k}^{q^{i-1}} \\ y_n^{q^{i+1}} & y_{n-1}^{q^{i+1}} & \cdots & y_{n+1-k}^{q^{i+1}} \\ \vdots & \vdots & & \vdots \\ y_n^{q^k} & y_{n-1}^{q^k} & \cdots & y_{n+1-k}^{q^k} \end{pmatrix},$$

the lemma yields

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j+n+1} u_{i-j}^{q^{\min\{i,j\}}} \cdot c_{k-1,i} \cdot d_{n-k,j} = c_{k,k} \cdot d_{n+1-k,n+1-k}.$$

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Now express $c_{k,i}$ in $d_{k,i}$ in terms of the f_i and g_i.

The Borel group B_n

$B_n(\mathbb{F}_q) = \{$upper triangular matrices$\}$. Set

$$\widetilde{f}_i = \prod_{h \in B_n(\mathbb{F}_q)\cdot x_i} h, \qquad \widetilde{g}_i = \prod_{h \in U_n(\mathbb{F}_q)\cdot y_{n-i+1}} h \qquad (i = 1, \ldots, n),$$

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Other groups

Observation: For $G \in \{U_n(\mathbb{F}_q), B_n(\mathbb{F}_q)\}$, $\mathbb{F}_q[V \oplus V^*]^G$ is generated by $\mathbb{F}_q[V]^G$, $\mathbb{F}_q[V^*]^G$, and some u_i's. Does this generalize?

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Conjecture: $\mathbb{F}_q[V \oplus V^*]^{\mathsf{GL}_n(\mathbb{F}_q)}$ is generated by the Dickson invariants in the x_i and in the y_i, and by $u_{1-n}, \ldots, u_{n-1}$.
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The corresponding statement is not true for $\mathsf{SL}_2(\mathbb{F}_3)$ (and apparently for no $\mathsf{SL}_n(\mathbb{F}_q)$).