Degree bounds for separating invariants

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(joint work with Hanspeter Kraft)
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- $K[V]^G$ invariant ring, natural grading $K[V]^G = \bigoplus_{d=0}^{\infty} K[V]_d^G$.

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- Noether's bound: G finite, $p \not| |G| \Rightarrow \beta(G) \leq |G|$
  (Fleischmann, Fogarty 2000/2001,
  Noether 1916 for $p = 0$ or $|G| < p$)

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- $\beta(D_{2n}) = n + 1$.
- Determination of $\beta(G)$ for some small groups.

Sharpenings and some more results

- Domokos, Hegedus, Sezer: G finite and $p \not| |G| \Rightarrow$

$$\beta(G) \leq \begin{cases} \frac{3}{4}|G| & \text{for } |G| \text{ even and } G \text{ non cyclic} \\ \frac{5}{8}|G| & \text{for } |G| \text{ odd and } G \text{ non cyclic} \end{cases}$$

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- Richman: G finite and $p||G| \Rightarrow \beta(G) = \infty$.
- Bryant, Derksen, Kemper: G a linear algebraic group. Then

$$\beta(G) < \infty \Leftrightarrow |G| < \infty \text{ and } p \not| |G|.$$

Separate instead of generate!

Separating sets (Derksen, Kemper)

$S \subseteq K[V]^G$ separating $:\Leftrightarrow$ For all $x, y \in V$ we have:
if $f(x) = f(y)$ for all $f \in S \Rightarrow f(x) = f(y)$ for all $f \in K[V]^G$.
Theorem: There always exists a finite separating set.

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- Draisma, Kemper, Wehlau: G finite $\Rightarrow$

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- find (improved) upper and lower bounds!

Relative Degree bounds

Theorem

$H \leq G$ closed of finite index $(G : H) < \infty$, V an H-module $\Rightarrow$

$$\beta_{\text{sep}}(H, V) \leq \beta_{\text{sep}}(G, KG \otimes_{KH} V), \quad \text{i.e. } \beta_{\text{sep}}(H) \leq \beta_{\text{sep}}(G)$$

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Theorem

$N \trianglelefteq G$ closed of finite index $(G : N) < \infty \Rightarrow$

$$\beta_{\text{sep}}(G) \leq \beta_{\text{sep}}(G/N)\beta_{\text{sep}}(N).$$

E.g. for finite groups G, H:

$$\beta_{\text{sep}}(G \rtimes H) \leq \beta_{\text{sep}}(G)\beta_{\text{sep}}(H).$$

p-groups and cyclic groups

MAGMA and $\beta_{\text{sep}}(G) = \beta_{\text{sep}}(G, V_{\text{reg}})$

$p = 2: \quad \beta_{\text{sep}}(S_3) = \beta_{\text{sep}}(S_3, V_{\text{reg}}) = 4.$

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Corollary: $|G| = p^k m, (p, m) = 1 \Rightarrow \beta_{\text{sep}}(G) \geq p^k.$

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Dihedral groups, the Hilbert ideal

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Corollary of Proofs:

G a p-group, or a cyclic group, or $G = D_{2p^r}$. There exists a V such that the Hilbert ideal $K[V]^G_+ K[V]$ is not generated by invariants of degree $< |G|$.

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- Recall: $\beta_{\text{sep}}(D_{2p^r}) = 2p^r$.
- Conjecture: $\text{char}(K) = 2$, $p \geq 3$ prime, $\Rightarrow \beta_{\text{sep}}(D_{2p}) = p + 1$

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Definitions

Basic $\mathbb{G}_a$-actions

$\mathbb{G}_a$ acts on the $n+1$ dimensional vector space $V_n = \langle X_0, \ldots, X_n \rangle_K$

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Frobenius powers

For any G-module U, define its p^m*th Frobenius power*

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Ex.: $\quad F^{p^m}(V_n) = \langle X_0^{p^m}, \ldots, X_n^{p^m} \rangle_K.$

Lower bounds for the additive group

Theorem

Assume $\mathrm{char}(K) = p > 0$. Then

$$\beta_{\mathrm{sep}}(\mathbb{G}_a, V_1 \oplus F^{p^n}(V_1)) = p^n + 1$$

$$\Rightarrow \quad \beta_{\mathrm{sep}}(\mathbb{G}_a) = \infty.$$

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Assume $\mathrm{char}(K) = 0$. Then

$$\beta_{\mathrm{sep}}(\mathbb{G}_a, V_1 \oplus V_n) \geq n + 1$$

$$\Rightarrow \quad \beta_{\mathrm{sep}}(\mathbb{G}_a) = \infty.$$

Roberts' isomorphism

Let $\langle X, Y\rangle$ be the natural representation of SL_2. For any SL_2-module V, there is the degree decreasing Roberts'-isomorphism

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Theorem

Robert's isomorphism maps SL_2-separating sets to $\mathbb{G}_a$-separating sets. In particular,

$$\beta_{\mathrm{sep}}(\mathrm{SL}_2) = \infty.$$

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- Set $u_{12} := X_1 Y_2 - X_2 Y_1$, $u_1 = X_1 Y - X Y_1$, $u_2 := X_2 Y - X Y_2$.

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- Set $u_{12} := X_1 Y_2 - X_2 Y_1$, $u_1 = X_1 Y - X Y_1$, $u_2 := X_2 Y - X Y_2$.
- $S := \{X_1, X_2, X_1 u_{12}, X_2 u_{12}\} \subseteq K[V]^{\mathbb{G}_a}$ is a separating set.

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Warning

In general, ismorphisms of algebras do not take separating sets to separating sets!

- Take $V^* := \langle X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle$ and $W := V^* \oplus \langle X, Y \rangle^*$.
- Set $u_{12} := X_1 Y_2 - X_2 Y_1$, $u_1 = X_1 Y - X Y_1$, $u_2 := X_2 Y - X Y_2$.
- $S := \{X_1, X_2, X_1 u_{12}, X_2 u_{12}\} \subseteq K[V]^{\mathbb{G}_a}$ is a separating set.
- Though $\Phi : K[W]^{\mathrm{SL}_2} \to K[V]^{\mathbb{G}_a}$ is an isomorphism,

$$\Phi^{-1}(S) = \{u_1, u_2, u_1 u_{12}, u_2 u_{12}\} \subseteq K[W]^{\mathrm{SL}_2}$$

is not a separating set!

Unipotent groups

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Theorem

G any infinite unipotent group (of arbitrary characteristic) $\Rightarrow$

$$\beta_{\mathrm{sep}}(G) = \infty$$

Semisimple groups

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Corollary (of unipotent & semisimple case)

Let G be a linear algebaic group. Then

$$\beta_{\text{sep}}(G) < \infty \Leftrightarrow |G| < \infty.$$

Thank You!