Rings of Invariants and Varieties of Representations

Dr James Shank, University of Kent, June 2010

Introduction and Notation

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If G is finite then $\mathbb{F}[V]^G$ is a finitely generated algebra.

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However, if p divides $|G|$ (the modular case) there may be infinitely many isomorphism classes of n-dimensional $\mathbb{F}G$-modules.

Example: $\mathbb{F} = \overline{\mathbb{F}_p}$, $G = \mathbb{Z}/p \times \mathbb{Z}/p =< g_1, g_2 >$, $n = 2$. Define

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$$Ny := \prod_{g \in G} y\rho(g) = \prod_{a,b \in \mathbb{F}_p} (y + (a + b\lambda)x) = y^{p^2} + xf(x, y, \lambda).$$

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Since $\lambda \in \mathbb{F}_p$ iff $\lambda^p = \lambda$, specialisation gives a generating set as long as we avoid roots of $t^p - t$.

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X is determined by the relations $\rho(r_j)$, which are polynomials in the entries of the matrices, and is thus a subvariety of $GL_n(\mathbb{F})^r$.

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For each element of $Aut(G)\backslash X/GL_n(\mathbb{F})$, there is a corresponding isomorphism class of subrings of $\mathbb{F}[x_1, \ldots, x_n]$, i.e., for $\varphi \in Aut(G)$ and $\sigma \in GL_n(\mathbb{F})$,

$$\mathbb{F}[x_1, \ldots, x_n]^{\rho(G)} \cong \mathbb{F}[x_1, \ldots, x_n]^{\sigma^{-1}\rho(\varphi(G))\sigma}.$$

Example: $\mathbb{F} = \overline{\mathbb{F}_p}$, $G = \mathbb{Z}/p \times \mathbb{Z}/p =< g_1, g_2 >$, $n = 3$. Define

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Furthermore, $\mathbb{F}[x, y, z]^{\rho(g_1)}$ is the hypersurface generated by x,

$$\begin{aligned} d &= y^2 - x(2z + y), \\ N_1 &= \prod_{a \in \mathbb{F}_p} (y + ax) = y^p - yx^{p-1}, \\ N_2 &= \prod_{a \in \mathbb{F}_p} \left(z + ay + \binom{a}{2} x\right) = z^p + \ldots, \end{aligned}$$

subject to a relation $d^p - N_1^2 + 2x^p N_2 + f(x, d)$.

Writing $\Delta = \rho(g_2) - 1$, we have

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1. The generic case: Calculate over $\mathbb{F}(t, s)$ and specialise t to λ and s to μ. The ring of invariants is a hypersurface with generators in degrees $1, p, p+2, p^2$ and a relation in degree $p(p+2)$. The specialisation gives the correct ring of invariants for $\lambda^p - \lambda \neq 0$ and $\lambda^2 - \lambda - 2\mu \neq 0$.

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2. $\mu = \binom{\lambda}{2}$: Then d is invariant. Calculate over $\mathbb{F}(t)$ and specialise t to λ. The ring of invariants is a hypersurface with generators in degrees $1, 2, p^2, p^2$ and a relation in degree $2p^2$. The specialisation gives the correct ring of invariants for $\lambda^p - \lambda \neq 0$.

3. $\lambda^p = \lambda$: Then N_1 is invariant. By replacing g_2 with $g_1^{-\lambda} g_2$, we can assume $\lambda = 0$. Calculate over $\mathbb{F}(s)$ and specialise s to μ. The ring of invariants is a hypersurface with generators in degrees $1, p, p+1, p^2$ and a relation in degree $p(p+1)$. The specialisation gives the correct ring of invariants for $\mu \neq 0$.

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4. $\lambda^p = \lambda$ and $\mu = \binom{\lambda}{2}$: The action is not faithful. In fact

$$\mathbb{F}[x, y, z]^{\rho(G)} = \mathbb{F}[x, y, z]^{\rho(g_1)}.$$

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Remarks:

- These are *a posteriori* observations.
- Can we develop a framework to identify the equations that describe the generic case without computing the ring of invariants?

In the $n = 2$ case, it is possible to embed the family of representations in a larger representation: $V(\lambda) \subset W$, giving $\mathbb{F}[W] \xrightarrow{\pi} \mathbb{F}[V(\lambda)]$.

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The long exact sequence coming from group cohomology gives

$$\cdots \to \mathbb{F}[W]^G \to \mathbb{F}[V(\lambda)]^G \to H^1(G, \ker(\pi)) \to H^1(G, \mathbb{F}[W]) \to \cdots$$

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Choose the embedding so that $\pi(x_3) = y$, $\pi(x_2) = x$ and $\pi(x_1) = cx$ where $c = (\lambda - \alpha)/(\alpha^p - \alpha)$.

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To verify π is a G-map:

$$\pi(x_3\Delta_2) = \pi(\alpha x_2 + (\alpha^p - \alpha)x_1) = (\alpha + (\alpha^p - \alpha)c)x = \lambda x.$$

It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with

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However, if $\lambda \in \mathbb{F}_p$ then $y^p - x^{p-1}y \in \mathbb{F}[V(\lambda)]^G \setminus \mathrm{im}(\pi^G)$,

It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with

$$N(x_3) = \prod_{g \in G} x_3 g = x_3^{p^2} + \cdots$$

and that $\ker(\pi) = (x_1 - cx_2)\mathbb{F}[W]$.

Thus $I := \ker(\pi)^G = (x_1 - cx_2)\mathbb{F}[W]^G$.

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I am still looking for the analog of W for the three variable case.