Modular Invariant Theory of the Cyclic Group via Classical Invariant Theory

David Wehlau

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Outline

- Modular RepresentationTheory of C_p.
- Representation Theory of $\mathrm{SL}_2(\mathbb{C})$
- The Connection
- Shank's Conjecture
- Proof of the Conjecture
- Some Consequences

Modular Representation Theory of C_p

Let $\mathbb{F}$ be any field of characteristic $p > 0$ and let C_p denote the cyclic group of order p. We fix a generator σ of C_p.

Suppose V is an indecomposable representation of C_p defined over $\mathbb{F}$. Consider the Jordan Normal Form of the matrix of σ in $\mathrm{GL}(V)$. This is the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

where $1 \leq n \leq p$.

1. We get p inequivalent indecomposable representations: $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_p$

2. $V_n^{C_p} \cong V_1$ for all n.

3. $\Delta := \sigma - 1$

4. $\mathrm{Tr}(f) = \sum_{i=0}^{p-1} \sigma^i \cdot f = (\sigma - 1)^{p-1} \cdot f = \Delta^{p-1}(f)$ where $\Delta = \sigma - 1$.
   Thus a non-zero invariant h is a transfer if and only if $h \in V_p^{C_p}$.

5. $\mathrm{Norm}(f) = \prod_{i=0}^{p-1} \sigma^i \cdot f$.

Known Modular Invariants of C_p

1. $\mathbb{F}_p[\bigoplus^m V_2]^{C_p}$
2. $\mathbb{F}_p[V_3]^{C_p}$
3. $\mathbb{F}_p[V_4]^{C_p}$
4. $\mathbb{F}_p[V_5]^{C_p}$
5. $\mathbb{F}_p[V_2 \oplus V_3]^{C_p}$
6. $\mathbb{F}_p[V_3 \oplus V_3]^{C_p}$
7. $\mathbb{F}_p[V_2 \oplus V_2 \oplus V_3]^{C_p}$

Representation Theory of $\mathrm{SL}_2(\mathbb{C})$

Representations of $\mathrm{SL}_2(\mathbb{C})$

1. R_1 denotes the defining two dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ with fixed basis $\{X, Y\}$.
2. $R_d := \mathrm{S}^d(R_1) = \mathrm{span}\{X^d, X^{d-1}Y, \ldots, Y^d\}$.
3. Every irreducible $\mathrm{SL}_2(\mathbb{C})$ representation is isomorphic to R_d for some d.
4. Classical invariant theorists studied not just the ring of invariants $\mathbb{C}[W]^{\mathrm{SL}_2(\mathbb{C})}$ of a representation W but also $\mathbb{C}[R_1 \oplus W]^{\mathrm{SL}_2(\mathbb{C})}$, the *ring of covariants of* W.

The Connection

Robert's Isomorphism (1861)

$$\begin{array}{rcl} \mathbb{C}[R_1 \oplus W]^{\mathrm{SL}_2(\mathbb{C})} & \longrightarrow & \mathbb{C}[W]^U \\ f(\cdot,\cdot) & \mapsto & f(X,\cdot) \end{array}$$

where

$$\begin{array}{rcl} U & = & \{\tau \in \mathrm{SL}_2(\mathbb{C}) \mid \tau \cdot X = X\} \\ & = & \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{C} \right\}. \end{array}$$

Robert's Isomorphism (1861)

$$\begin{array}{ccc} \mathbb{C}[R_1 \oplus W]^{\mathrm{SL}_2(\mathbb{C})} & \xrightarrow{\sim} & \mathbb{C}[W]^U \\ f(\cdot,\cdot) & \mapsto & f(X,\cdot) \end{array}$$

where

$$\begin{aligned} U &= \{\tau \in \mathrm{SL}_2(\mathbb{C}) \mid \tau \cdot X = X\} \\ &= \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{C} \right\}. \end{aligned}$$

The Integers

U contains the subgroup

$$U \supset \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}$$

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U contains the subgroup

$$U \supset \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\} = \mathbb{Z}$$

The fact that $\mathbb{Z}$ is a dense subgroup of U in the Zariski topology implies

$\mathbb{C}[W]^U = \mathbb{C}[W]^Z$.

Note that the action of $\mathbb{Z}$ is generated by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Define

- $M_n := R_{n-1}(\mathbb{Q})$
- $L_n := R_{n-1}(\mathbb{Z}) = M_n(\mathbb{Z})$
- If W is an $\mathrm{SL}_2(\mathbb{C})$-representation, $M = W(\mathbb{Q})$ and $L = W(\mathbb{Z})$

If $W = \oplus_{i=1}^{t} R_{n_i-1}$ then $M = \oplus_{i=1}^{t} M_{n_i}$ and $L = \oplus_{i=1}^{t} L_{n_i}$.

The element $1 \in \mathbb{Z} \subset U$ acts on R_{n-1}, M_n and L_n via the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

$$\begin{aligned}
\mathbb{C}[R_1 \oplus W]^{\mathrm{SL}_2(\mathbb{C})} &\cong \mathbb{C}[W]^U = \mathbb{C}[W]^{\mathbb{Z}} \\
&\cong \mathbb{Q}[M]^{\mathbb{Z}} \otimes_{\mathbb{Q}} \mathbb{C} \\
&\cong (\mathbb{Z}[L]^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\
&\cong \mathbb{Z}[L]^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}
\end{aligned}$$

$$\begin{array}{ccc}
S^\bullet(L^*) = \mathbb{Z}[L] & \supset & \mathbb{Z}[L]^{\mathbb{Z}} \\
\downarrow \ (\mathrm{mod}\ p) & & \downarrow \ (\mathrm{mod}\ p) \\
S^\bullet(V^*) = \mathbb{F}_p[V] & \supset & \mathbb{F}_p[V]^{C_p}
\end{array}$$

Elements of $\mathbb{F}_p[V]^{C_p}$ which lie in the image of $\mathbb{Z}[L]^{\mathbb{Z}}$ under reduction modulo p are called *integral* invariants.

Shank's Conjecture

$\mathbb{F}[V]^{C_p}$ is generated by

1. Certain specific norms (one for each summand of V).
2. Certain transfers (finitely many).
3. A finite set of integral invariants.

Proof of the conjecture

The Tensor Algebra

$\mathbb{F}_p[V_n]_d = S^d(V_n^*)$ is a summand of $\bigotimes^d V^*$. We consider $\bigotimes^d V_n$.

$$
\begin{array}{ccccc}
S^d(M_n) & \subset & (\bigotimes^d M_n) & \cong & M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_s} \\
\cup & & \cup & & \cup \\
S^d(L_n) & \subset & (\bigotimes^d L_n) & & L_{n_1} \oplus L_{n_2} \oplus \cdots \oplus L_{n_s} \\
\downarrow \pmod{p} & & \downarrow \pmod{p} & & \downarrow \pmod{p} \\
S^d(V_n) & \subset & (\bigotimes^d V_n) & \cong & V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_s}
\end{array}
$$

We need to show that the co-kernel of the map on invariants is spanned by transfers for $d < p$.

Example ($p = 7$)

$$
\begin{array}{ccccc}
S^4(M_5) & \subset & \bigotimes^4 M_5 & \cong & \begin{array}{c} 5M_1 \oplus 12M_3 \oplus 16M_5 \oplus 17M_7 \\ \oplus 15M_9 \oplus 10M_{11} \oplus 6M_{13} \oplus 3M_{15} \oplus M_{17} \end{array} \\
\cup & & \cup & & \cup \\
S^4(L_5) & \subset & \bigotimes^4 L_5 & & \begin{array}{c} 5L_1 \oplus 12L_3 \oplus 16L_5 \oplus 17L_7 \\ \oplus 15L_9 \oplus 10L_{11} \oplus 6L_{13} \oplus 3L_{15} \oplus L_{17} \end{array} \\
\downarrow (\mathrm{mod}\ p) & & \downarrow (\mathrm{mod}\ p) & & \downarrow (\mathrm{mod}\ p) \\
S^4(V_5) & \subset & \bigotimes^4 V_5 & \cong & 2V_1 \oplus 3V_3 \oplus V_5 \oplus 87V_7
\end{array}
$$

Three Representation Rings

Let $m \leq n$. We compare:

- $R_{m-1} \otimes R_{n-1}$
- $M_m \otimes M_n$
- $V_m \otimes V_n$

$$R_m \otimes R_n \cong R_{n-m} \oplus R_{n-m+2} \oplus \cdots \oplus R_{m+n}$$

$$R_{m-1} \otimes R_{n-1} \cong R_{n-m+1} \oplus R_{n-m+3} \oplus \cdots \oplus R_{m+n-1}$$

Jordan Normal Form

$$M_m \otimes M_n \cong M_{n-m+1} \oplus M_{n-m+3} \oplus \cdots \oplus M_{m+n-1}$$

Roth (1934)

Aiken (1934) (Littelwood 1936)

Markus and Robinson (1975)

Brualdi (1985)

Representation ring of C_p

If $1 \leq m \leq n \leq p$,

$V_m \otimes V_n \cong$

$$\begin{cases} V_{n-m+1} \oplus V_{n-m+3} \oplus \cdots \oplus V_{m+n-1}, & \text{if } m+n \leq p+1; \\ V_{n-m+1} \oplus V_{n-m+3} \oplus \cdots \oplus V_{2p-m-n-1} \oplus (\oplus^{m+n-p} V_p), & \text{if } m+n \geq p. \end{cases}$$

$$
\begin{array}{ccc}
M_m \otimes M_n & \cong & M_{n-m+1} \oplus M_{n-m+3} \oplus \cdots \oplus M_{m+n-1} \\
\cup & & \cup \\
L_m \otimes L_n & & L_{n-m+1} \oplus L_{n-m+3} \oplus \cdots \oplus L_{m+n-1} \\
\downarrow \pmod p & & \downarrow \pmod p \\
V_m \otimes V_n & \cong & V_{n-m+1} \oplus V_{n-m+3} \oplus \cdots \oplus V_{n+m-1}
\end{array}
$$

if $m + n \leq p + 1$.

$$
\begin{array}{ccc}
M_m \otimes M_n & \cong & M_{n-m+1} \oplus M_{n-m+3} \oplus \cdots \oplus M_{m+n-1} \\
\cup & & \cup \\
L_m \otimes L_n & & L_{n-m+1} \oplus L_{n-m+3} \oplus \cdots \oplus L_{m+n-1} \\
\downarrow \pmod{p} & & \downarrow \pmod{p} \\
V_m \otimes V_n & \cong & V_{n-m+1} \oplus V_{n-m+3} \oplus \cdots \oplus V_{2p-m-n-1} \oplus (\oplus^{m+n-p} V_p)
\end{array}
$$

if $m + n \geq p$.

It suffices to prove that L_r surjects onto V_r here
for all $r \leq 2p - m - n - 1$ when $m \leq n \leq p$.

Consider $m = 3$ and $n = 4$.

$$\begin{array}{ccc}
M_3 \otimes M_4 & \cong & M_2 \oplus M_4 \oplus M_6 \\
\cup & & \cup \\
L_3 \otimes L_4 & & L_2 \oplus L_4 \oplus L_6 \\
\downarrow \pmod p & & \downarrow \pmod p \\
V_3 \otimes V_4 & \cong & V_2 \oplus V_4 \oplus V_6
\end{array}$$

if $p \geq 7$.

If $m = 3$, $n = 4$ and $p = 5$.

$$\begin{array}{ccc}
M_3 \otimes M_4 & \cong & M_2 \oplus M_4 \oplus M_6 \\
\cup & & \cup \\
L_3 \otimes L_4 & & L_2 \oplus L_4 \oplus L_6 \\
\downarrow \pmod{p} & & \downarrow \pmod{p} \\
V_3 \otimes V_4 & \cong & V_2 \oplus V_5 \oplus V_5
\end{array}$$

$M_3 \otimes M_4 \cong \mathbb{Q}[s]/(s^3) \otimes \mathbb{Q}[t]/t^4 \cong \mathbb{Q}[s,t]/(s^3, t^4)$.
$M_3 \otimes M_4 \cong M_2 \oplus M_4 \oplus M_6$.
Δ acts via multiplication by $(1+s)(1+t) - 1 = s + t + st$.

$$\begin{array}{c|ccccc}
0 & & & 1 & & \\
1 & & s & & t & \\
2 & s^2 & & st & & t^2 \\
3 & s^2t & & st^2 & & t^3 \\
4 & & s^2t^2 & & st^3 & \\
5 & & & s^2t^3 & &
\end{array}$$

$$\begin{array}{c|ccccc}
0 & & & \bullet & & \\
1 & & \bullet & & \bullet & \\
2 & \bullet & & \bullet & & \bullet \\
3 & \bullet & & \bullet & & \bullet \\
4 & & \bullet & & \bullet & \\
5 & & & \bullet & &
\end{array}$$

$M_3 \otimes M_4 \cong \mathbb{Q}[s]/(s^3) \otimes \mathbb{Q}[t]/t^4 \cong \mathbb{Q}[s,t]/(s^3, t^4)$.
$M_3 \otimes M_4 \cong M_2 \oplus M_4 \oplus {\color{red} M_6}$.
Δ acts via multiplication by $(1+s)(1+t) - 1 = s + t + st$.

$$
\begin{array}{lccccccc}
0 & & & & 1 & & & \\
1 & & & s & & t & & \\
2 & & s^2 & & st & & t^2 & \\
3 & & s^2t & & st^2 & & t^3 & \\
4 & & & s^2t^2 & & st^3 & & \\
5 & & & & s^2t^3 & & &
\end{array}
$$

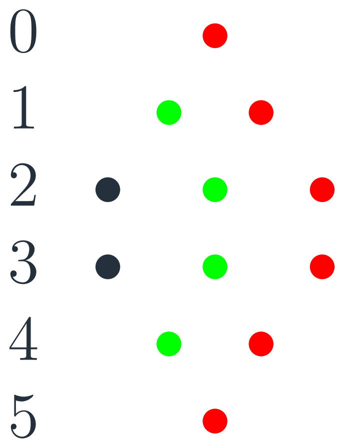
$$
\begin{array}{lccccc}
0 & & & {\color{red}\bullet} & & \\
1 & & \bullet & & {\color{red}\bullet} & \\
2 & \bullet & & \bullet & & {\color{red}\bullet} \\
3 & \bullet & & \bullet & & {\color{red}\bullet} \\
4 & & \bullet & & {\color{red}\bullet} & \\
5 & & & {\color{red}\bullet} & &
\end{array}
$$

$M_3 \otimes M_4 \cong \mathbb{Q}[s]/(s^3) \otimes \mathbb{Q}[t]/t^4 \cong \mathbb{Q}[s,t]/(s^3,t^4)$.

$M_3 \otimes M_4 \cong M_2 \oplus M_4 \oplus M_6$.

Δ acts via multiplication by $(1+s)(1+t) - 1 = s + t + st$.

0		1	
1	s	t	
2	s^2	st	t^2
3	s^2t	st^2	t^3
4	s^2t^2	st^3	
5		s^2t^3	



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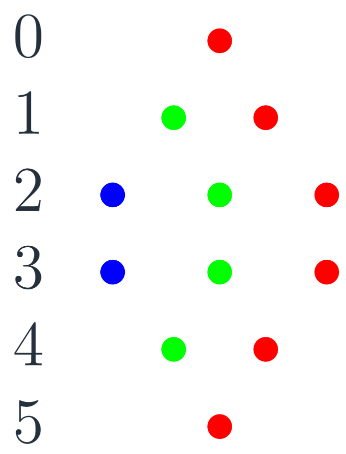
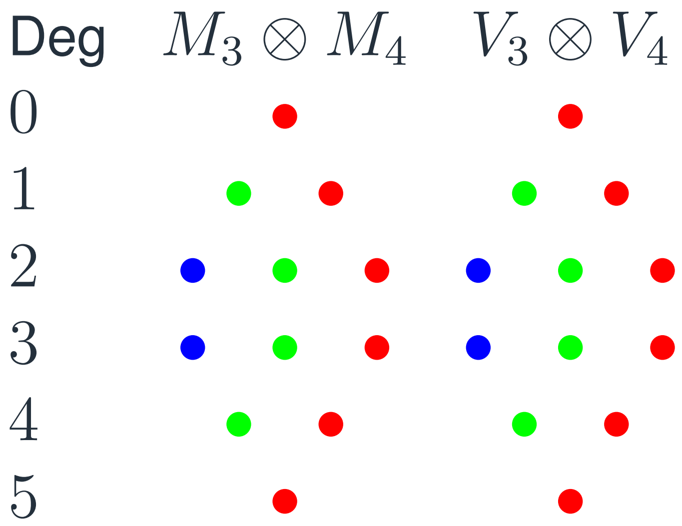
$M_3 \otimes M_4 \cong M_2 \oplus M_4 \oplus M_6$.
Δ acts via multiplication by $(1+s)(1+t)-1 = s+t+st$.

$$\begin{array}{llll}
0 & & 1 & \\
1 & & s \quad t & \\
2 & s^2 & st & t^2 \\
3 & s^2t & st^2 & t^3 \\
4 & & s^2t^2 \quad st^3 & \\
5 & & s^2t^3 &
\end{array}$$



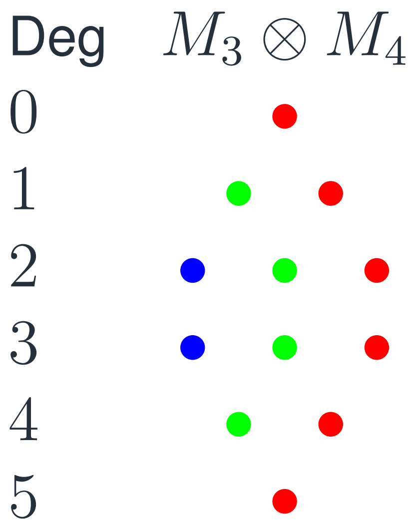
$$M_3 \otimes M_4 \cong \mathbb{Q}[s,t]/(s^3,t^4) \cong {\color{blue}M_2} \oplus {\color{green}M_4} \oplus {\color{red}M_6}.$$

If $p \geq 7$:
$$V_3 \otimes V_4 \cong \mathbb{F}_p[s,t]/(s^3,t^4) \cong {\color{blue}V_2} \oplus {\color{green}V_4} \oplus {\color{red}V_6}.$$



$$M_3 \otimes M_4 \cong \mathbb{Q}[s,t]/(s^3,t^4) \cong M_2 \oplus M_4 \oplus M_6.$$

If $p = 5$:
$$V_3 \otimes V_4 \cong \mathbb{F}_p[s,t]/(s^3,t^4) \cong V_2 \oplus V_5 \oplus V_5.$$



$$M_3 \otimes M_4 \cong \mathbb{Q}[s,t]/(s^3,t^4) \cong M_2 \oplus M_4 \oplus M_6.$$

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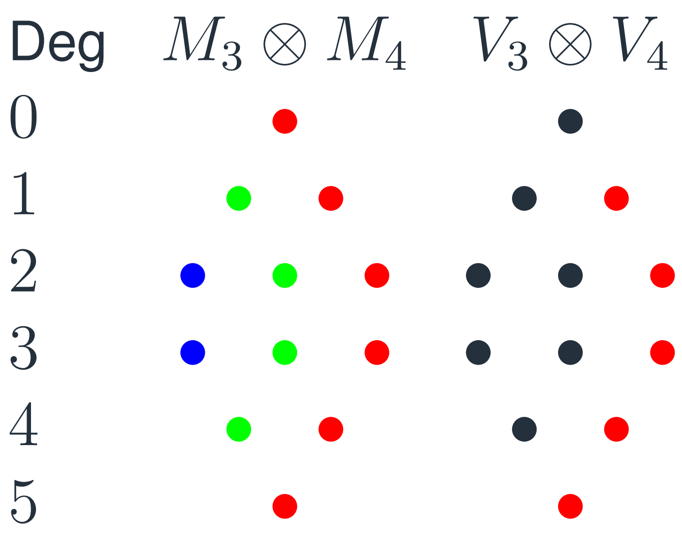
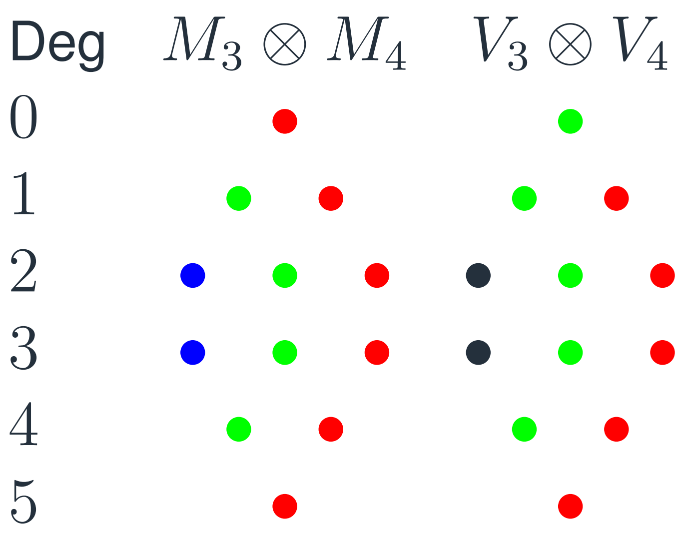


$$M_3 \otimes M_4 \cong \mathbb{Q}[s,t]/(s^3,t^4) \cong M_2 \oplus M_4 \oplus M_6.$$

If $p = 5$:
$$V_3 \otimes V_4 \cong \mathbb{F}_p[s,t]/(s^3,t^4) \cong V_2 \oplus V_5 \oplus V_5.$$



$$M_3 \otimes M_4 \cong \mathbb{Q}[s,t]/(s^3,t^4) \cong M_2 \oplus M_4 \oplus M_6.$$

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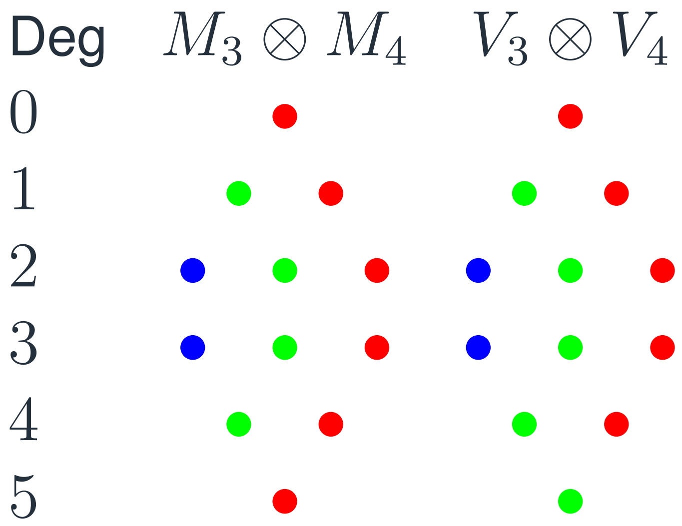
$$V_3 \otimes V_4 \cong \mathbb{F}_p[s,t]/(s^3,t^4) \cong V_2 \oplus V_5 \oplus V_5.$$



If $p \geq 7$

$M_3 \otimes M_4$	M_6	$1 \mapsto s + t + st \mapsto s^2 + 2st + t^2 + 2s^2t + 2st^2 + s^2t^2$ $\mapsto 3s^2t + 3st^2 + t^3 + 6s^2t^2 + 3st^3 + 3s^2t^3$ $\mapsto 6s^2t^2 + 4st^3 + 12s^2t^3 \mapsto 10s^2t^3 \mapsto 0$
	M_4	$3s - 2t \mapsto 3s^2 + st - 2t^2 + 3s^2t - 2st^2$ $\mapsto 4s^2t - st^2 - 2t^3 + 2s^2t^2 - 4st^3 - 2s^2t^3$ $\mapsto 3s^2t^2 - 3st^3 - 3s^2t^3 \mapsto 0$
	M_2	$3s^2 - 2st + t^2 + 2t^3 \mapsto s^2t - st^2 + t^3 - 2s^2t^2 + 3st^3 \mapsto 0$

$V_3 \otimes V_4$	V_6	$1 \mapsto s + t + st \mapsto s^2 + 2st + t^2 + 2s^2t + 2st^2 + s^2t^2$ $\mapsto 3s^2t + 3st^2 + t^3 + 6s^2t^2 + 3st^3 + 3s^2t^3$ $\mapsto 6s^2t^2 + 4st^3 + 12s^2t^3 \mapsto 10s^2t^3 \mapsto 0$
	V_4	$3s - 2t \mapsto 3s^2 + st - 2t^2 + 3s^2t - 2st^2$ $\mapsto 4s^2t - st^2 - 2t^3 + 2s^2t^2 - 4st^3 - 2s^2t^3$ $\mapsto 3s^2t^2 - 3st^3 - 3s^2t^3 \mapsto 0$
	V_2	$3s^2 - 2st + t^2 + 2t^3 \mapsto s^2t - st^2 + t^3 - 2s^2t^2 + 3st^3 \mapsto 0$

Explicit Decompositions

If $p = 5$

$M_3 \otimes M_4$	M_6	$1 \mapsto s + t + st \mapsto s^2 + 2st + t^2 + 2s^2t + 2st^2 + s^2t^2$ $\mapsto 3s^2t + 3st^2 + t^3 + 6s^2t^2 + 3st^3 + 3s^2t^3$ $\mapsto 6s^2t^2 + 4st^3 + 12s^2t^3 \mapsto 10s^2t^3 \mapsto 0$
	M_4	$3s - 2t \mapsto 3s^2 + st - 2t^2 + 3s^2t - 2st^2$ $\mapsto 4s^2t - st^2 - 2t^3 + 2s^2t^2 - 4st^3 - 2s^2t^3$ $\mapsto 3s^2t^2 - 3st^3 - 3s^2t^3 \mapsto 0$
	M_2	$3s^2 - 2st + t^2 + 2t^3 \mapsto s^2t - st^2 + t^3 - 2s^2t^2 + 3st^3 \mapsto 0$
$V_3 \otimes V_4$	V_5	$1 \mapsto s + t + st \mapsto s^2 + 2st + t^2 + 2s^2t + 2st^2 + s^2t^2$ $\mapsto 3s^2t + 3st^2 + t^3 + 6s^2t^2 + 3st^3 + 3s^2t^3$ $\mapsto 6s^2t^2 + 4st^3 + 12s^2t^3 \mapsto 0 = 10s^2t^3$
	V_5	$s \mapsto s^2 + st + s^2t \mapsto 2s^2t + st^2 + 2s^2t^2$ $\mapsto 3s^2t^2 + st^3 + 3s^2t^3 \mapsto 4s^2t^3 \mapsto 0$
	V_2	$3s^2 - 2st + t^2 + 2t^3 \mapsto s^2t - st^2 + t^3 - 2st^3 + 3s^2t^2 \mapsto 0$

Some Consequences

Some New Modular Invariants

1. $\mathbb{F}[V_2 \oplus V_4]^{C_p}$ Covariants of $R_1 \oplus R_3$ were computed classically.
2. $\mathbb{F}[V_3 \oplus V_4]^{C_p}$ Covariants of $R_2 \oplus R_3$ were computed classically.
3. $\mathbb{F}[\oplus^m V_3]^{C_p}$ Covariants of $\oplus^m R_2$ were computed classically.
4. $\mathbb{F}[\oplus^m V_4]^{C_p}$ Covariants of $\oplus^m R_3$ were computed by Schwarz(1987).
5. $\mathbb{F}[V_6]^{C_p}$ Covariants of the quintic were computed by Gordan in 1869.
6. $\mathbb{F}[V_7]^{C_p}$ Covariants of the sextic were computed by Gordan in 1869.
7. $\mathbb{F}[V_8]^{C_p}$ Covariants of the septic were computed by Bedratyuk(2009).
8. $\mathbb{F}[V_9]^{C_p}$ Covariants of the octic were computed by von Gall in 1880.

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