Exceptional low genus actions of finite groups

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Monodromy
Permutation groups
Exceptional low genus actions
Tools

Further work
Zeroing in on $E(0)$
Actions and tuples

Work in progress with

Robert Guralnick
Corneliiu Hoffman
Kay Magaard
Path lifting

\(X, Y\) compact Riemann surfaces
\(f : X \to Y\) holomorphic, degree \(n\) (generically \(n\) to 1)
\(\mathcal{B} = \{y \in Y \mid |f^{-1}(y)| < n\} = \{y_1, \ldots, y_r\}\) “Branch points”
\(Y_0 = Y \setminus \mathcal{B}\) \(y_0 \in Y_0\)
Path lifting

Let $X, Y$ be compact Riemann surfaces. Let $f : X \to Y$ be holomorphic, with degree $n$ (generically $n$ to 1). Let $B = \{y \in Y \mid |f^{-1}(y)| < n\} = \{y_1, \ldots, y_r\}$ be its branch points. Define $Y_0 = Y \setminus B$ and $y_0 \in Y_0$.

For a closed path $\lambda$ in $Y_0$ from $y_0$ to $y_0$, path lifting gives a permutation of $f^{-1}(y_0)$, independent of homotopy type. So $f$ induces a map $\pi_1(Y_0, y_0) \to S_n$. [Symmetric group]
The monodromy group

The image of $\pi_1(Y_0)$ in $S_n$ is $\text{Mon}(f)$, the \textit{monodromy group} of the cover $f$.

Case of interest: $Y = P^1(\mathbb{C})$, the Riemann sphere and $X$ is connected, so $\text{Mon}(f)$ is a transitive subgroup of $S_n$ with generators $\sigma_1, \ldots, \sigma_r$ satisfying the relation $\sigma_1 \sigma_2 \cdots \sigma_r = 1$. 
Examples

1. $X = Y = P^1(\mathbb{C})$, $f : z \mapsto z^n$, $\mathcal{B} = \{0, \infty\}$.

$y_0 = 1$. $f^{-1}(y_0) = \{e^{2\pi ik/n}, k = 0, 1, \ldots, n - 1\}$

$\text{Mon}(f)$ is cyclic of order $n$, generated by $\alpha \mapsto \zeta \alpha$, $\zeta = e^{2\pi i/n}$.
Examples

1. \(X = Y = P^1(\mathbb{C}), f : z \rightarrow z^n, B = \{0, \infty\}\).
   \(y_0 = 1, f^{-1}(y_0) = \{e^{2\pi ik/n} \mid k = 0, 1, \ldots, n-1\}\)

   \(\text{Mon}(f)\) is cyclic of order \(n\), generated by \(\alpha \mapsto \zeta \alpha\),
   \(\zeta = e^{2\pi i/n}\).

2. \(X = Y = P^1(\mathbb{C}), f : X \rightarrow (1 - n)X^n + nX^{n-1}, B = \{0, 1, \infty\}\) of respective multiplicities \(n - 1, 2, n\).

   The monodromy group is \(S_n\), generated by an \((n - 1)\)-cycle and a 2-cycle with product an \(n\)-cycle.
Riemann’s Existence Theorem

Fix a finite subset \( B = \{ b_1, \ldots, b_r \} \) of \( P^1(\mathbb{C}) \). Then there is a 1–1 correspondence

\[
\{ \text{Isomorphism classes of maps } F : X \to P^1(\mathbb{C}) \text{ of degree } n \text{ with branch points } B \} \leftrightarrow \{ \text{Conjugacy classes of } r\text{-tuples } (\sigma_1, \ldots, \sigma_r) \text{ of permutations in } S_n \text{ such that } \sigma_1 \cdots \sigma_r = 1 \text{ and } \langle \sigma_i \rangle \text{ is transitive} \}
\]

Moreover if \( \sigma_i \) has cycle structure \( (m_1, \ldots, m_k) \), then there are \( k \) preimages \( u_1, \ldots, u_k \) of \( b_i \) in the corresponding cover \( F : X \to Y \), with \( \text{mult}_{u_j}(F) = m_j \) for each \( j \).

Generating systems

$(x_1, x_2, \ldots, x_r)$ is a generating system for the group $G$ provided $x_i \neq 1$, $i = 1, \ldots, r$ and

$$G = \langle x_1, x_2, \ldots, x_r \rangle$$  \hspace{2cm} (1)

$$x_1 \cdot x_2 \cdot \ldots \cdot x_r = 1$$  \hspace{2cm} (2)
Generating systems

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\[ G = \langle x_1, x_2, \ldots, x_r \rangle \quad (1) \]

\[ x_1 \cdot x_2 \cdot \ldots \cdot x_r = 1 \quad (2) \]

Note: (1) and (2) provide a presentation of $\pi_1(P^1(\mathbb{C}) \setminus \{b_1, \ldots, b_r\})$. 
X of genus $g$, $Y = P^1(\mathbb{C})$, $f$ of degree $n$, $G = \text{Mon}(f) \Rightarrow$

$$\sum \text{Ind}(x_i) = 2(n + g - 1) \quad (3)$$

for some generating system $(x_i)$

Conversely, $G \leq S_n$, transitive, (3) $\Rightarrow$

$$\exists \ X \text{ of genus } g, \ f : X \to P^1(\mathbb{C}) \text{ with } \text{Mon}(f) \cong G.$$
For $x \in S_n$,

$$\text{Ind}(x) = n - |\text{Orb}(\langle x \rangle)|$$

where $\text{Orb}(\langle x \rangle)$ is the set of orbits of $\langle x \rangle$.

$$\text{Ind}(x) = \dim [V, \tilde{x}]$$

where $\tilde{x}$ is the image of $x$ acting on the vector space $V$ with basis $\{e_1, \ldots, e_n\}$.

Equivalently, $\text{Ind}(x)$ is the smallest non-negative integer $r$ such that $x = \tau_1 \tau_2 \cdots \tau_r$ for transpositions $\tau_i$ in $S_n$. 
Application

$L_3(2)$ cannot be generated by elements $x, y, z$ of order 2 such that $x$ and $z$ commute.

Proof: Suppose $x, y, z$ are involutions with $xz = zx$. If $x = z$ then $\langle x, y, z \rangle$ is dihedral. Otherwise, $xz$ is an involution. Consider the 5-tuple $(x, z, xz, y, y)$ in $L_3(2)$. In the natural action on 7 points all involutions have permutation index 4. If $L_3(2) = \langle x, y, z \rangle$ then this 5-tuple would be a generating system for a monodromy group of a surface of genus -1.
Let $\mathcal{E}(0)$ be the set of all noncyclic, non-alternating composition factors of monodromy groups of maps $P^1(\mathbb{C}) \to P^1(\mathbb{C})$.

Replace domain by $X$ of genus 1, 2, ... to define $\mathcal{E}(g)$ for all non-negative $g$.

Conjecture (1990): $\mathcal{E}(g)$ is finite for every $g$. 
Some key steps in establishing the GT-conjecture

Guralnick, Thompson (1990) “Finite Groups of Genus Zero” set out basic objective and reduced it to the primitive, almost simple case using T. Shih (1991), as well as other work of Guralnick and others.

Liebeck, Saxl (1990) general bounds for fprs for groups of Lie type \( \Rightarrow q \) is bounded for Lie type groups over \( F_q \) of given genus.

Liebeck, Shalev (1999) asymptotic bounds for non-subspace actions of classical groups reduced to subspace actions.

F, Magaard (1998) fpr bounds in terms of eigenspace codimensions for classical groups.

F, Magaard (2001): \( \mathcal{E}(0) \) is finite.
Cauchy-Frobenius Formula

\[ H \leq S_n \Rightarrow \]

\[ |\text{Orb}(H)| = n - \frac{1}{|H|} \sum_{h \in H} \text{Fix}(h) \]

Consequence: \( x \in S_n, \, o(x) = d \Rightarrow \)

\[ \text{Ind}(x) = \frac{d - 1}{d} n - \frac{\sum_{y \in \langle x \rangle} \# \text{Fix}(y)}{d} \]
Let \( x \) be a generating system and set \( d_i = o(x_i) \).
For \( d = (d_1, \ldots, d_r) \), (the \textit{signature} of \( x \)) set

\[
A(d) = \sum_i \frac{d_i - 1}{d_i}
\]

The equation derived from the CF and RH formulas can be written as

\[
\sum \frac{1}{o(\sigma)} \left( \sum_{\tau \in \langle \sigma \rangle^\#} \text{fpr}(\tau) \right) = A(d) - 2 - \frac{g - 1}{n}
\]
Interpreting the left side as \([A(d)]\) times a weighted average of fixed point ratios, this provides, when \(g\) is fixed and \(n\) gets large, a lower bound for the average fixed point ratio of a set of non-identity elements of \(G\).

In particular, if \(g = 0\), then

\[
\text{fpr}(\tau) \geq \frac{A(d) - 2}{A(d)}
\]

for some non-identity element \(\tau\) in \(G\).
Properties of $A(d)$

If $\mathbf{x}$ is a generating system for a group $G$ and $d$ is the signature of $\mathbf{x}$ then one of the following holds.

1. $G$ is solvable
2. $G \cong A_5$
3. $A(d) \geq 85/42$

In particular,

$$\frac{A(d) - 2}{A(d)} \geq \frac{1}{85}$$

for all relevant groups.
Scott’s Theorem (1977) on linear groups

*If G is a linear group acting on V with \([G, V] = V\) and \([G, V^*] = V^*\) and \(\sigma_1, \ldots, \sigma_r\) is a generating system for G then \(\sum v(\sigma_i) \geq 2 \dim V\) where \(v(\sigma)\) is the codimension of \(C_V(\sigma)\).*

Combine with generators and relations argument to bound the total contribution of large fixed point ratios
But wait, there’s more!

Quote from Guralnick, Thompson’s 1990 paper:

*hope remains that $\mathcal{E}(0)$ may be explicitly determined.*
Relevant results

Magaard (1993): Genus 0 sporadic group actions

F, Guralnick, Magaard (2001): Genus 0 groups of Lie rank 1

Lawther, Liebeck, Seitz (2002): Fixed point bounds in exceptional groups

F, Guralnick, Magaard (2002): Reduction to point actions or special circumstances for classical groups

Guralnick, Shareshian (2007): Symmetric and alternating group actions with many branch points (genus grows rapidly except in special cases)

F, Guralnick, Magaard (2014): Degree bound \((n < 10^4)\) for primitive classical groups of genus at most 2
$\mathcal{E}(0)$, the set of exceptional composition factors of monodromy groups of genus zero consists of the following 45 isomorphism types:

- $L_2(q), 7 \leq q \leq 43, q \neq 9, 23$
- $L_2(64)$
- $L_3(3), L_3(4), L_3(5), L_3(7)$
- $L_4(3), L_4(4), L_5(2), L_5(3), L_6(2)$
- $U_3(3), U_3(4), U_3(5)$
- $Sp_4(3), Sp_4(4), Sp_4(5)$
- $Sp_6(2), Sp_8(2)$
- $O_8^+(2), O_8^-(2)$
- $Sz(8)$
- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Co_3, J_1, J_2, HS$
Pinning things down further

Knowing the composition factors, determine

1. Groups up to isomorphism
2. (Primitive) actions
3. Types of generating systems (tuples of conjugacy classes)
4. Generating tuples up to equivalency
Braiding

Two generating systems are equivalent if one can be transformed into the other by a series of the following operations:

1. Conjugation by elements of $G$. $x \in G$ fixed:
   \[(x_1, x_2, \ldots, x_r) \mapsto (x_1^x, x_2^x, \ldots, x_r^x),\]

2. Braid operations, e.g.
   \[(x_1, x_2, x_3) \mapsto (x_2, x_1^{x_2}, x_3)\]
   \[(x_1, x_2, x_3) \mapsto (x_1^{x_1 x_2}, x_2^{x_1 x_2}, x_3)\]

Note that the second braid operation is the same as conjugation by $x_3^{-1}$. The analogous property does not hold when $r > 3$!
How many inequivalent actions?

For almost simple groups of type $L_2(q)$

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